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# ALTERNATING CURRENT AND RESONANT CIRCUITS

## ABSTRACT

The frequency responses of first and second order circuits are characterized and compared to theory in this experiment. The voltage drop across components of an RC series circuit, an RL series circuit, and an RLC series circuit were measured and normalized to the alternating input voltage. Response when driven at the resonant frequency was observed for the RLC circuit, and the quality factor was calculated within 15% of the theoretical value. The observed frequency response closely resembled theory for all circuits, with errors arising mostly from non-idealities of components and sources. The pulse response of the RLC circuit was also observed; the natural response was underdamped, and the time constant and frequency of oscillation were measured within 8.1% and 2.9% of the theoretical values, respectively.

## INTRODUCTION

The study of alternating current is one of the most practical topics in the classical field of electricity and magnetism. AC power systems are the world standard, with the voltage difference between the terminals of a standard outlet being v(t) = 170 cos 377 t.[1] One of the most useful techniques for analyzing AC circuits is phasor representation, which was developed around 1893 by Charles Steinmetz for General Electric.[2] A phasor is a vector in the complex plane that can represent a voltage or current given as a function of time without losing any important information. The total impedance, **Z**, presented by the passive elements (resistors, capacitors, and inductors) of a circuit, is defined as the ratio **Z** = **V**/**I**, and is also a phasor with a real and imaginary component.[2]

In the first part of this experiment, two first order circuits are examined. In a series RC circuit at low frequencies, the voltage drop is mostly across the capacitor. However, since the voltage drop across a capacitor cannot change instantaneously, the majority of the voltage drop is across the resistor at very high frequencies. In a series RL circuit at low frequencies, the inductor is very much like a short circuit, and most of the voltage drops across the resistor, while at high frequencies the majority of the voltage drop is across the inductor.[2]

In the second part of this experiment, a series RLC circuit is examined. RLC circuits and other resonant circuits that can select a small range of frequencies are the basis for many electronic devices, allowing electromagnetic waves to be generated, transmitted, and received, as in a standard radio. The majority of the voltage drop is across the resistor only in a range of frequencies during which the effects of the inductor and the capacitor cancel each other out. The size of this range is quantified in the quality factor, Q, of a circuit. Circuits with a high Q have most of the voltage drop across the resistor for a very small range of frequencies. A radio receiver usually requires a Q of several hundred.[1]

The pulse response of an RLC circuit is also measured in this experiment. Given an initial voltage, such as a pulse or the turn-on of a DC source, energy is transferred back and forth between the capacitor (where it is stored in electric fields) and the inductor (where it is stored in magnetic fields).[1] When the resistance is low, as in this experiment, the response is underdamped, taking the form of a damped sinusoidal function.

## THEORY

An alternating voltage source v(t) = V cos (ùt + ö) can be represented in phasor form as **V** = $Ve^{jö}$ = V∠ö, where **V** is a vector on the complex plane of length V at angle ö. The phasor voltage and current are related by the formula **V** = **ZI**, where **Z** is the total impedance, with $\mathbf{Z}_R$ = R, $\mathbf{Z}_L$ = jùL, and $\mathbf{Z}_C$ = 1/jùC for sinusoidal waveforms.[2]

Using voltage divider, the ratio of the voltage across the resistor to the source voltage in an RC series circuit can be written

$$\frac{\mathbf{V}_R}{\mathbf{V}_{in}} = \frac{\mathbf{Z}_R}{\mathbf{Z}_{total}} = \frac{\mathbf{Z}_R}{\mathbf{Z}_C + \mathbf{Z}_R} = \frac{R}{1/j\omega C + R} = \frac{j\omega RC}{1 + j\omega RC}$$

The magnitude and phase angle of $\mathbf{V}_R/\mathbf{V}_{in}$ can then be represented as follows:

$$\left|\frac{\mathbf{V}_R}{\mathbf{V}_{in}}\right| = \frac{\omega RC}{\sqrt{1+(\omega RC)^2}} \qquad \angle \frac{\mathbf{V}_R}{\mathbf{V}_{in}} = 90^o - \tan^{-1}(\omega RC)$$

Since the output voltage drop across the resistor is close to the input voltage at high frequencies, this is known as a high pass filter.

Similarly, the ratio $\mathbf{V}_C/\mathbf{V}_{in}$ can be found using voltage divider, and then its magnitude and phase are known:

$$\frac{\mathbf{V}_C}{\mathbf{V}_{in}} = \frac{\mathbf{Z}_C}{\mathbf{Z}_{total}} = \frac{\mathbf{Z}_C}{\mathbf{Z}_C + \mathbf{Z}_R} = \frac{1/j\omega C}{1/j\omega C + R} = \frac{1}{1 + j\omega RC}$$

$$\left|\frac{\mathbf{V}_{\mathrm{C}}}{\mathbf{V}_{\mathrm{in}}}\right| = \frac{1}{\sqrt{1+(\omega \mathrm{RC})^2}} \qquad \angle \frac{\mathbf{V}_{\mathrm{C}}}{\mathbf{V}_{\mathrm{in}}} = -\tan^{-1}(\omega \mathrm{RC})$$

An RC circuit with the output voltage measured across the capacitor is known as a low pass filter, since the output voltage is similar to the input voltage at low frequencies.

In the RL circuit,

$$\frac{\mathbf{V}_{\mathrm{R}}}{\mathbf{V}_{\mathrm{in}}} = \frac{\mathbf{Z}_{\mathrm{R}}}{\mathbf{Z}_{\mathrm{total}}} = \frac{\mathbf{Z}_{\mathrm{R}}}{\mathbf{Z}_{\mathrm{R}}+\mathbf{Z}_{\mathrm{L}}} = \frac{\mathrm{R}}{\mathrm{R}+\mathrm{j}\omega \mathrm{L}} = \frac{1}{1+\mathrm{j}\omega \mathrm{L}/\mathrm{R}}$$

$$\left|\frac{\mathbf{V}_{\mathrm{R}}}{\mathbf{V}_{\mathrm{in}}}\right| = \frac{1}{\sqrt{1+(\mathrm{j}\omega \mathrm{L}/\mathrm{R})^2}} \qquad \angle \frac{\mathbf{V}_{\mathrm{R}}}{\mathbf{V}_{\mathrm{in}}} = -\tan^{-1}(\omega \mathrm{L}/\mathrm{R})$$

$$\frac{\mathbf{V}_{\mathrm{L}}}{\mathbf{V}_{\mathrm{in}}} = \frac{\mathbf{Z}_{\mathrm{L}}}{\mathbf{Z}_{\mathrm{total}}} = \frac{\mathbf{Z}_{\mathrm{L}}}{\mathbf{Z}_{\mathrm{R}}+\mathbf{Z}_{\mathrm{L}}} = \frac{\mathrm{j}\omega \mathrm{L}}{\mathrm{R}+\mathrm{j}\omega \mathrm{L}} = \frac{\mathrm{j}\omega \mathrm{L}/\mathrm{R}}{1+\mathrm{j}\omega \mathrm{L}/\mathrm{R}}$$

$$\left|\frac{\mathbf{V}_{\mathrm{L}}}{\mathbf{V}_{\mathrm{in}}}\right| = \frac{\omega \mathrm{L}/\mathrm{R}}{\sqrt{1+(\omega \mathrm{L}/\mathrm{R})^2}} \qquad \angle \frac{\mathbf{V}_{\mathrm{L}}}{\mathbf{V}_{\mathrm{in}}} = 90^{\mathrm{o}} - \tan^{-1}(\omega \mathrm{L}/\mathrm{R})$$

Thus, the RL circuit acts as a low pass filter when output is measured across the resistor, and as a high pass filter when output is measured across the inductor.

Finally, for an RLC series circuit,

$$\frac{\mathbf{V}_{\mathrm{R}}}{\mathbf{V}_{\mathrm{in}}} = \frac{\mathbf{Z}_{\mathrm{R}}}{\mathbf{Z}_{\mathrm{R}}+\mathbf{Z}_{\mathrm{L}}+\mathbf{Z}_{\mathrm{C}}} = \frac{\mathrm{R}}{\mathrm{R}+\mathrm{j}(\omega \mathrm{L}-1/\omega \mathrm{C})} = \frac{1}{1+(\mathrm{j}/\mathrm{R})(\omega \mathrm{L}-1/\omega \mathrm{C})}$$

$$\left|\frac{\mathbf{V}_{\mathrm{R}}}{\mathbf{V}_{\mathrm{in}}}\right| = \frac{1}{\sqrt{1+[(1/\mathrm{R})(\omega \mathrm{L}-1/\omega \mathrm{C})]^2}} \qquad \angle \frac{\mathbf{V}_{\mathrm{R}}}{\mathbf{V}_{\mathrm{in}}} = -\tan^{-1}\left(\frac{\omega \mathrm{L}-1/\omega \mathrm{C}}{\mathrm{R}}\right)$$

$$\frac{\mathbf{V}_{\mathrm{LC}}}{\mathbf{V}_{\mathrm{in}}} = \frac{\mathbf{Z}_{\mathrm{L}}+\mathbf{Z}_{\mathrm{C}}}{\mathbf{Z}_{\mathrm{R}}+\mathbf{Z}_{\mathrm{L}}+\mathbf{Z}_{\mathrm{C}}} = \frac{\mathrm{j}(\omega \mathrm{L}-1/\omega \mathrm{C})}{\mathrm{R}+\mathrm{j}(\omega \mathrm{L}-1/\omega \mathrm{C})} = \frac{(\omega \mathrm{L}-1/\omega \mathrm{C})^2+\mathrm{jR}(\omega \mathrm{L}-1/\omega \mathrm{C})}{\mathrm{R}^2+(\omega \mathrm{L}-1/\omega \mathrm{C})^2}$$

$$\left|\frac{\mathbf{V}_{\mathrm{LC}}}{\mathbf{V}_{\mathrm{in}}}\right| = \frac{\sqrt{(\omega \mathrm{L}-1/\omega \mathrm{C})^2}}{\sqrt{\mathrm{R}^2+(\omega \mathrm{L}-1/\omega \mathrm{C})^2}} \qquad \angle \frac{\mathbf{V}_{\mathrm{LC}}}{\mathbf{V}_{\mathrm{in}}} = \tan^{-1}\left(\frac{\mathrm{R}}{\omega \mathrm{L}-1/\omega \mathrm{C}}\right)$$

For a series RLC circuit, the frequency at which the imaginary part of the impedance is zero is called the resonant frequency, or ù$_{o}$. Since **Z** = R + j(ùL – 1/ùC), ù$_{o}$ = 1/ (LC), so at resonance **Z**$_{o}$ = R, and the entire voltage drop is across the resistor.

The quality factor, Q, of a circuit, is a dimensionless ratio associated with a band pass circuit, which can be defined and calculated for an RLC series circuit as follows:

$$Q = 2\pi \frac{\text{maximum energy stored}}{\text{energy dissipated/cycle}} = 2\pi \frac{LI_{rms}^2}{RI_{rms}^2 / f_o} = 2\pi f_o \frac{L}{R} = \omega_o \frac{L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Q can also be calculated from the half-power frequencies, ù$_1$ and ù$_2$, at which the magnitude $|\mathbf{V}_{out}/\mathbf{V}_{in}|$ = 1/ 2:

$$Q = \frac{\omega_o}{\omega_2 - \omega_1}$$

The more selective a band pass filter is, the higher the quality factor.[2]

When a pulse is applied to a series RLC circuit, the circuit behaves as if connected to a DC source with a switch closed when the pulse turns on. The particular or forced response of the circuit is zero, since after a long time the inductor will appear as a short circuit and the capacitor will appear as an open circuit to the DC source, so no current will flow and the entire voltage drop will be across the capacitor. The natural or homogenous response can be found using the impedance function. With $R_1$ representing the resistor and $R_2$ representing the extra resistance of the source and inductor, the impedance function looking in from terminals across $R_1$ can be written as follows:

$$Z(s) = \frac{R_1(R_2 + sL + 1/sC)}{R_1 + R_2 + sL + 1/sC} = \frac{s^2 R_1 L + R_1 R_2 s + R_1/C}{s^2 L + (R_1 + R_2)s + 1/C}$$

Using the poles of the Z(s) for the natural voltage,

$$s = -\frac{R_1 + R_2}{2L} \pm \sqrt{\left(\frac{R_1 + R_2}{2L}\right)^2 - \frac{1}{LC}}$$

When the resistance is low, the response will be underdamped, and s will equal –á ± jù, where the natural response is of the form $v_n(t)$ = A e$^{-át}$ sin(ùt + ö).

## EXPERIMENT

Three different circuits were analyzed in this experiment, all consisting of individual elements in series with a signal generator, which applied a sinusoidal voltage to each circuit. The first circuit contained a resistor and a capacitor, the second contained a resistor and an inductor, and the third contained a resistor, an inductor, and a capacitor. The components used in each circuit were measured with a precision meter.

Using an oscilloscope, the total voltage drop across each circuit and the drops across individual elements were separately measured. The peak-to-peak magnitude of each voltage drop and the time separation of the peaks of the two signals were recorded for a range of frequencies. In the first order circuits (the RC and RL circuits), the voltage drop across each element was measured, and in the RLC circuit, the voltage drop across the resistor ($V_R$) and the voltage drop across the combination of the inductor and capacitor ($V_{LC}$) were measured.

The response of the RLC circuit to a short pulse was also measured, and the oscilloscope data were downloaded to a computer for further analysis.

The data from all circuits in this experiment were analyzed using Kaleidograph.

## RESULTS

The following values were obtained when the components in each circuit were measured with the precision meter. For the RC series circuit, C = 258.5 pF and R = 1025 Ù. For the RL series circuit, L = 291.2 ìH, and R = 100 Ù. For the RLC series circuit, R = 10.72 Ù, C = 258.8 pF, and L = 289.8 ìH (with $R_L$ = 6.18 Ù). For the pulsed RLC series circuit, R = 47.31 Ù, C = 250.7 pF, and L = 295 ìH (with $R_L$ = 5.96 Ù). The last digit of each measurement is uncertain.

For the RC series circuit, the theoretical and calculated magnitude of $\mathbf{V}_{out}/\mathbf{V}_{in}$ is displayed in Figure 1, and the phase difference between $\mathbf{V}_{out}$ and $\mathbf{V}_{in}$ is displayed in Figure 2, where $\mathbf{V}_{out}$ is measured separately across the resistor and across the capacitor.

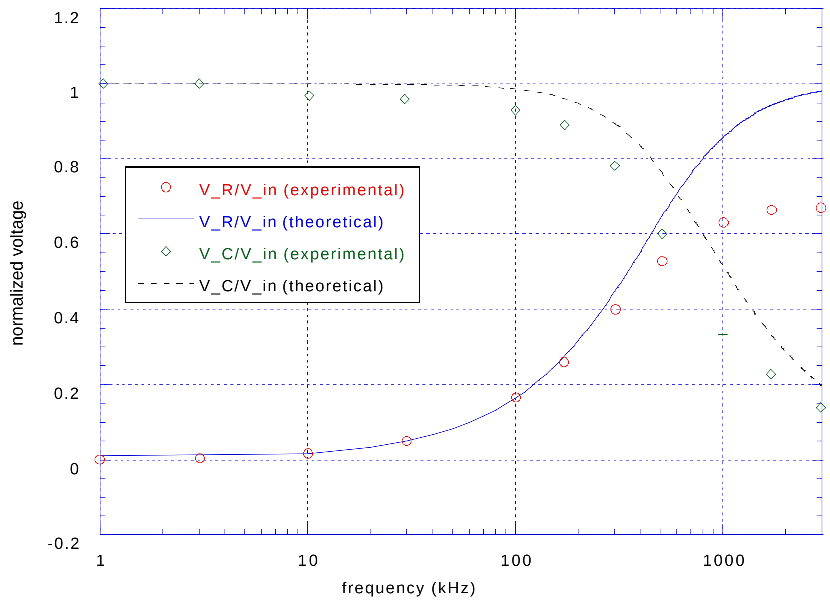


**Figure 1. Normalized magnitude of voltage drop across R and C in RC series circuit**

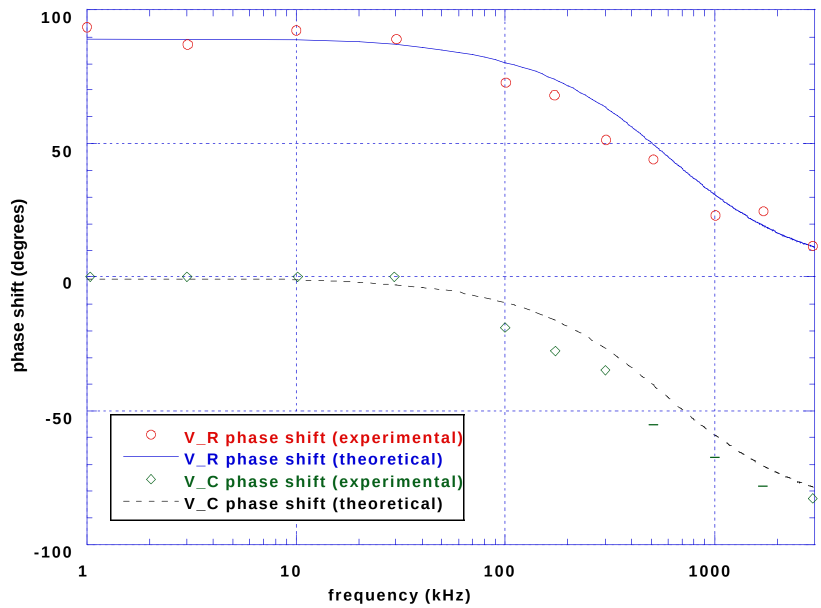


**Figure 2. Phase difference between input voltage and voltage across individual elements in RC series circuit**

For the RL series circuit, the theoretical and calculated magnitude of $\mathbf{V}_{out}/\mathbf{V}_{in}$ is displayed in Figure 3, and the phase difference between $\mathbf{V}_{out}$ and $\mathbf{V}_{in}$ is displayed in Figure 4, where $\mathbf{V}_{out}$ is measured separately across the resistor and across the inductor.

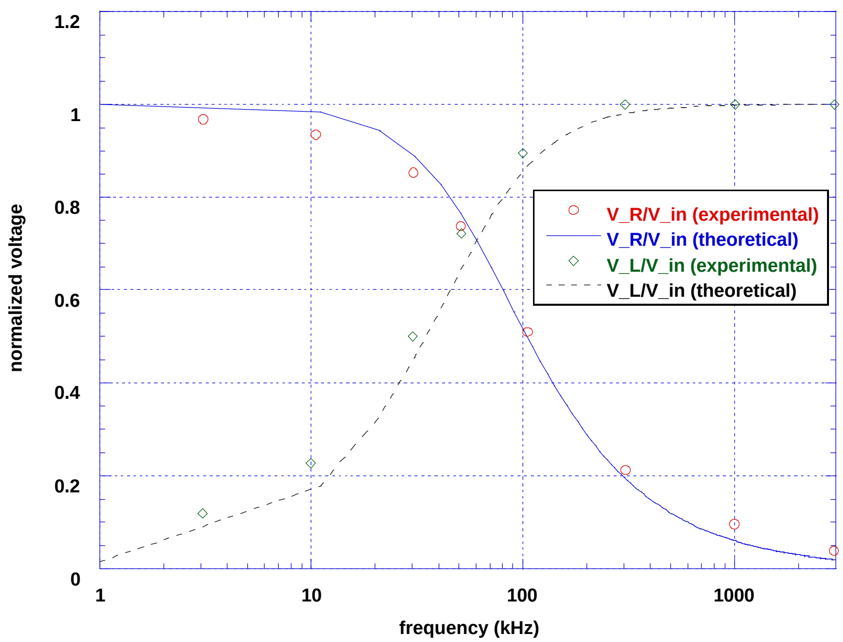


Figure 3. Normalized magnitude of voltage drop across R and L in RL series circuit

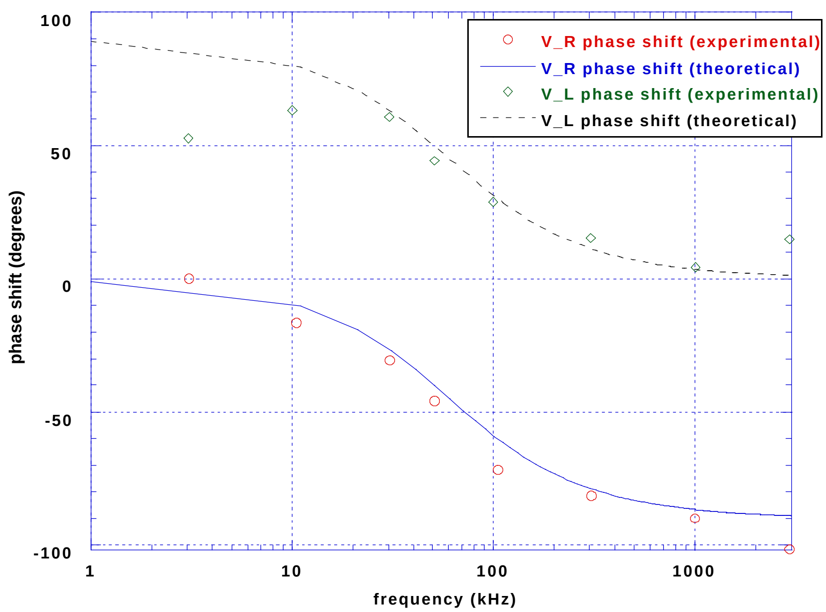


Figure 4. Phase difference between input voltage and voltage across individual elements in RL series circuit

For the RLC series circuit, the theoretical and calculated magnitude of $\mathbf{V}_{out}/\mathbf{V}_{in}$ is displayed in Figure 5, and the phase difference between $\mathbf{V}_{out}$ and $\mathbf{V}_{in}$ is displayed in Figure 6, where $\mathbf{V}_{out}$ is measured separately across the resistor and across the combination of capacitor and inductor.

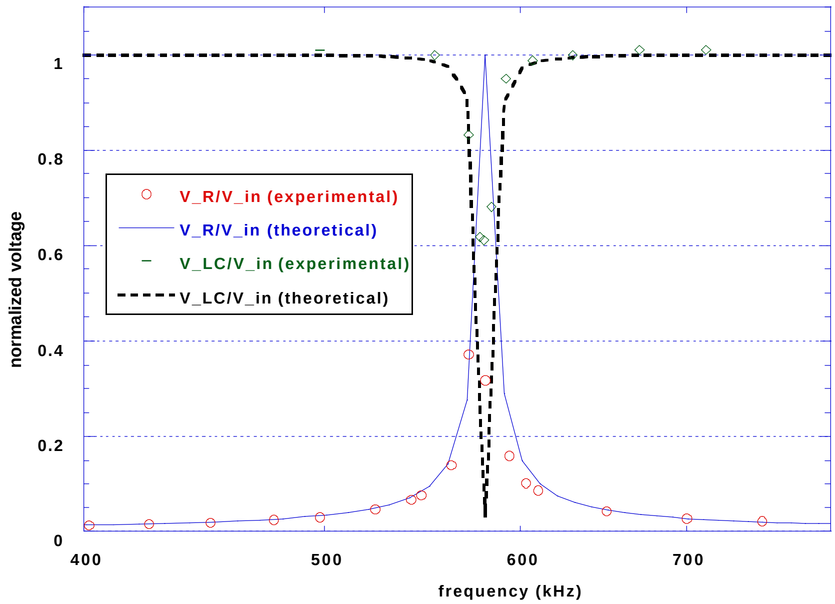


Figure 5. Normalized magnitude of voltage drop across R and LC in RLC series circuit

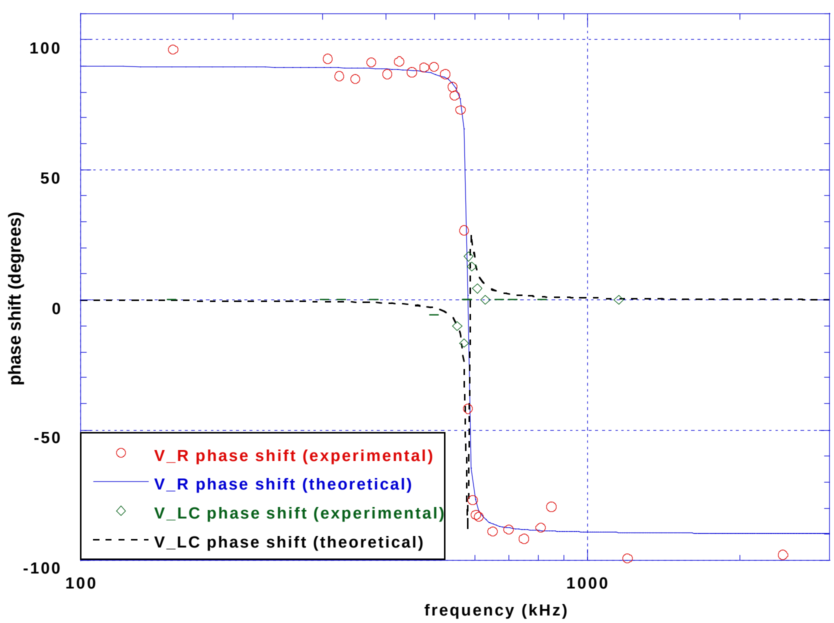


Figure 6. Phase difference between input voltage and voltage across R and LC in RLC series circuit

The resonant frequency, $f_o$, at which $|\mathbf{V}_R/\mathbf{V}_{in}|$ peaks and the phase of $\mathbf{V}_R/\mathbf{V}_{in}$ is zero degrees, falls somewhere between 572 kHz and 581 kHz, according to our measurements. By using $f_o$ = 1/(2ð LC), $f_o$ = 581 kHz, which is within the experimental measurement.

At $f_o$, the output voltage was at a maximum of 260 mV, so the voltage cursors were set 260/ 2 = 183 mV apart on the oscilloscope, and the half-power frequencies at these voltages were 560 kHz and 603 kHz. The quality factor of the circuit can then be calculated as

$$Q_{exp} = \frac{\omega_o}{\omega_2 - \omega_1} = \frac{f_o}{f_2 - f_1} = \frac{581}{43} = 13.5$$

This should equal the $Q_{theory}$ calculated using the measured component values:

$$Q_{theory} = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{10.72 + 6.18 + 50}\sqrt{\frac{289.8\times10^{-6}}{258.8\times10^{-12}}} = 15.8$$

The percent difference between $Q_{theory}$ and $Q_{exp}$ is 15%.

When we attempted to construct a phasor diagram of $\mathbf{V}_C$, $\mathbf{V}_L$, and $\mathbf{V}_{LC}$ at resonance, $V_{in}$ was observed to change depending on whether the oscilloscope was connected across the LC combination or just L. Then, with the f = 333 kHz, the following measurements were taken: $V_{in}$ = 1.98 V, $V_R$ = 0.04 V, $V_{LC}$ = 2.02 V, $V_L$ = 1.30 V, and $V_C$ = 3.80 V.

For the pulsed RLC circuit, a pulse of height 5.0 ± 0.1 V and width 9.8 ± 0.1 ì s was applied. The output voltage across the resistor during and after the pulse is displayed in Figure 7.

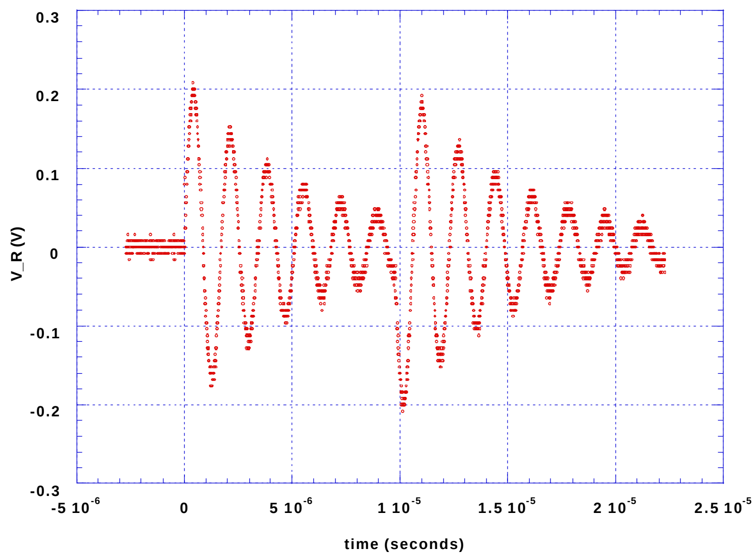


Figure 7. Response of resistor in pulsed RLC circuit. Pulse amplitude is 5.0 +/- 0.1 V, and pulse width is 9.8 microseconds.

The period for each cycle was found to be 1.72 ìs, so ù = 2ð/T = 3.57 x $10^6$ rad/s. When ù is calculated using the values of the components and the equation given in the theory section, ù = 3.67 x $10^6$ rad/s, a 2.9% difference from the experimental calculation.

The voltage and time at each peak and valley of oscillation were recorded using the oscilloscope cursors, and a decaying exponential was fit to these points in Kaleidograph to find ô = 5.24 ± 0.08 ìs, as seen in Figure 8.

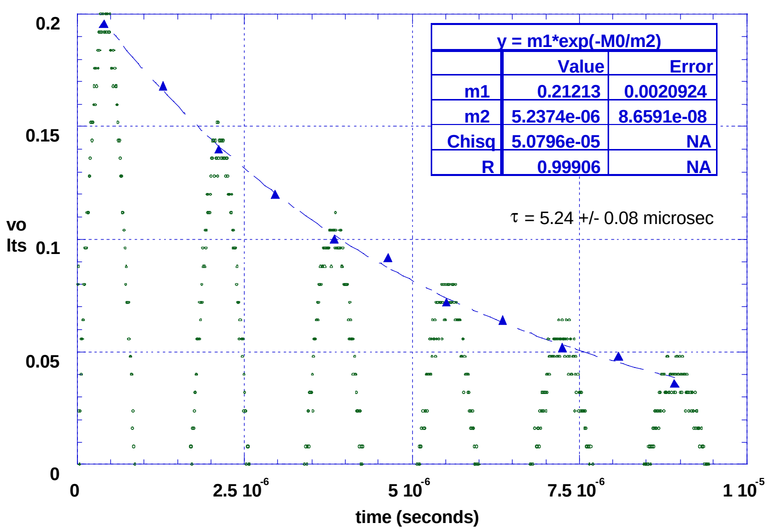


Figure 8. Calculation of RLC circuit time constant using the envelope of the output oscillations, which was obtained from recording the peaks and valleys on the oscilloscope. The time constant is m2.

Using the measured values of the components á = $R_{total}$/2L = 1.75 x $10^5$, so ô = 1/á = 5.7 ìs, an 8.1% difference from the value found from Kaleidograph.

## DISCUSSION

The relationship between voltage and current for capacitors and inductors is related by a derivative or integral, resulting in the fact that the voltage across a capacitor lags the current through it by $90^o$, and the voltage across an inductor leads the current through it by $90^o$. In a series RC or RL circuit, the phase of the current equals the phase of the voltage across the resistor. Thus, the phase of the voltage across the capacitor should be $90^o$ behind the phase of the voltage across the resistor in an RC circuit, which is what is observed in Figure 2. Similarly, the $\mathbf{V}_L/\mathbf{V}_{in}$ phase shift in an RL circuit should be $90^o$ ahead of the $\mathbf{V}_R/\mathbf{V}_{in}$ phase shift, as seen in Figure 4.

Since phasors are vectors in the complex plane, and since $\mathbf{V}_C/\mathbf{V}_{in} \perp \mathbf{V}_R/\mathbf{V}_{in}$ in the RC circuit and $\mathbf{V}_L/\mathbf{V}_{in} \perp \mathbf{V}_R/\mathbf{V}_{in}$ in the RL circuit, the Pythagorean Theorem states that in the RC circuit

$$(\mathbf{V}_C/\mathbf{V}_{in})^2 + (\mathbf{V}_R/\mathbf{V}_{in})^2 = 1$$

and in the RL circuit

$$(\mathbf{V}_L/\mathbf{V}_{in})^2 + (\mathbf{V}_R/\mathbf{V}_{in})^2 = 1.$$

Phasor diagrams of the RC and RL circuits and interesting frequencies are seen in Figures 9 and 10. The RC phasor diagram approximately represents our measurement at f = 172.5 kHz, and the RL phasor diagram approximately represents our measurement at f = 30.3 kHz.

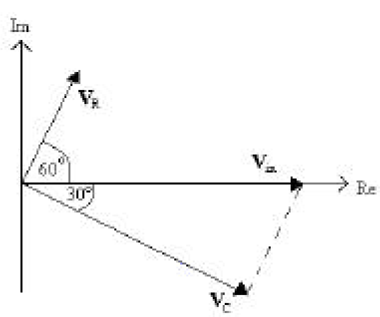


Figure 9. Phasor diagram of RC circuit at interesting frequency.

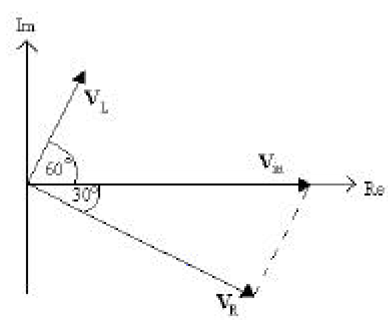


Figure 10. Phasor diagram of RL circuit at interesting frequency.

For the RLC circuit, an ideal phasor diagram at resonance, which occurs in our RLC circuit at $f_o$ = 581 kHz, is seen in Figure 11.

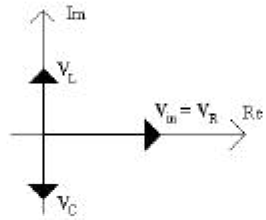

Figure 11. Phasor diagram of RLC series circuit at resonance.

The measurements taken at f = 333 Hz cannot be used to construct a phasor diagram, for the vectors cannot mathematically add up at any angles. It can be concluded that by attaching the oscilloscope to the circuit for measurements, it changes the circuit somehow.

Most of the experimental data appeared very close to theory for the frequency response graphs. The graph that most distinctly deviates from the theoretical curve is that of the phase shift across the inductor in the RL circuit; whereas the theoretical curve goes to $90^o$ at low frequencies, the experimental data points start curving downwards. Since the phase shift was calculated by dividing the time difference by the period, and the period is very large at very small frequencies, any errors in measuring the time shift are amplified. The $\mathbf{V}_L/\mathbf{V}_{in}$ phase shift could also be calculated by adding $90^o$ to the $\mathbf{V}_R/\mathbf{V}_{in}$ phase shift, which would yield a more accurate result.

The fairly large error in calculating the quality factor of the RLC circuit is probably due to the imprecision of the method used to find the half power frequencies. The difference between the frequencies could easily have been 37 Hz instead of 43 Hz, and then the experimental Q would have matched the theoretical one. Also, there may have been stray impedances and non-idealities in sources and cables that were not accounted for in the theoretical Q.

The response of the pulsed circuit very closely matched theory, as the difference between the theoretical and experimental time constant was 8.1%, and the difference between the theoretical and experimental frequency was 2.1%. It is interesting to note the response after the pulse turned off; it appears similar to the response during the pulse, except it starts going down instead of going up. During the duration of the pulse, a voltage difference is accumulating across the capacitor; if the pulse lasted a long time, the entire amplitude of the pulse voltage would drop across the capacitor, and no current would flow in the circuit. When the pulse turns off, the capacitor discharges, inducing a current in the opposite direction and a response across the resistor, which is similar (but opposite) to its response during the pulse.

Conclusions

These experiments illustrate some of the important concepts of alternating current and resonant circuits. The frequency responses of two first order circuits (series RC and RL) are examined, illustrating high pass and low pass filters. A second order resonant circuit (series RLC) is also examined. Just as simple harmonic oscillators in mechanics have a constant oscillation frequency, even as they are damped by forces such as friction, the charge in an RLC circuit oscillates with a constant frequency between the capacitor and inductor while the resistor damps the amplitude of oscillation, as was observed during the pulse response across the resistor. With output measured across the resistor, the RLC circuit acts as a band pass filter; the quality factor, which is inversely related to the width of the peak in the magnitude frequency response plot, was calculated. Experimental observations closely matched theory for all circuits, with errors arising from measurements and non-idealities in sources and components.

## References

1. Purcell, Edward M. *Electricity and Magnetism, Berkeley Physics Course Volume 2, Second Edition*. New York: McGraw-Hill, 1985.

2. Smith, Ralph J. and Richard C. Dorf. *Circuits, Devices and Systems, Fifth Edition*. New York: John Wiley & Sons, Inc, 1992.