The Lucas Circles of a Triangle

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product space), and strictly positive on A. We define $\delta(x, \varepsilon) = \text{dist}((x, x, \varepsilon), F)$; then $\delta : X \times]0, \infty[\to \mathbb{R}$ is Lipschitz continuous, with Lipschitz constant not larger than 1, and strictly positive since A contains the set $\{(x, x, \varepsilon) : x \in X, \varepsilon > 0\}$. If $x, y \in X$, and $|y - x| < \delta(x, \varepsilon)$ then (x, y, ε) belongs to A, hence $|f(y) - f(x)| < \varepsilon$. ∎

Remark A straightforward modification of this proof shows that δ also depends continuously on the function f, if the uniform topology is considered on the set of continuous functions from X to Y; see [**2**].

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The Lucas Circles of a Triangle

Antreas P. Hatzipolakis and Paul Yiu

It is easy to inscribe a square in a given triangle ABC. Construct a square externally on the side BC. Join the vertex A to the two remote vertices of the square. The intersections X_1, X_2 with BC are two vertices of the inscribed square. The perpendiculars to BC at these two points intersect AC and AB at the remaining two vertices Y_3 and Z_4; see Figure 1. The other two inscribed squares, $Y_1Y_2Z_3X_4$ and $Z_1Z_2X_3Y_4$, can be similarly constructed.

It is also easy to calculate the side lengths of the inscribed squares, by making use of the similarity (homothety) of triangles ABC and AZ_4Y_3. In [**1**, p. 458], we find an expression for the circumradius R_a of triangle AY_3Z_4, in terms of the side lengths a, b, c and the circumradius R of triangle ABC:

$$R_a = \frac{Rbc}{bc + 2aR}.$$

Similar expressions for R_b and R_c can be written down. Panakis attributes the circumcircles of triangles AY_3Z_4, BZ_3X_4, and CX_3Y_4 to the noted 19th century mathematician Édouard Lucas, and leaves as an easy exercise the following identity.

$$\frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{R_c} = \frac{3}{R} + \frac{2(a^2 + b^2 + c^2)}{abc}.$$

It is amazing that when we actually construct the circles, we find that they are mutually tangent to each other; see Figure 2. We are certain that Lucas would have known this, though it is not mentioned in [**1**]. Here is a proof.

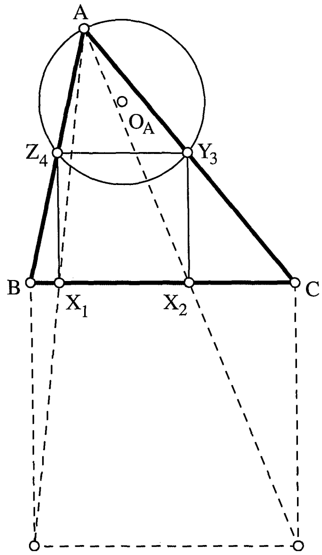

Figure 1.

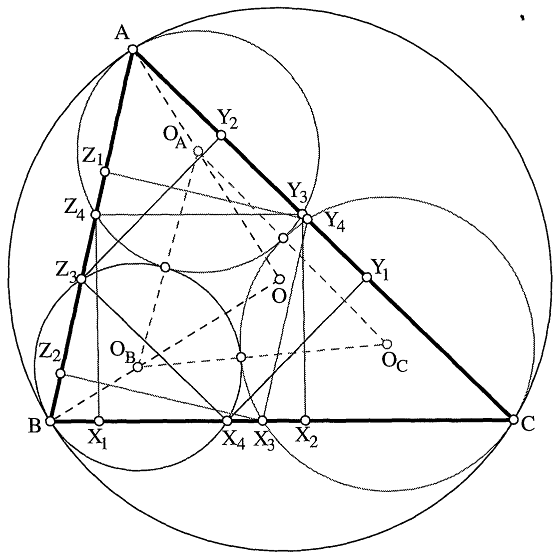

Figure 2.

Note that the similarity with center A and ratio AZ_4/AB takes triangle ABC and its circumcircle to triangle AZ_4Y_3 and its circumcircle. Since a similarity takes the center of a circle to the center of its image circle, the line joining the circumcenters O and O_A passes through the point A, and

$$|OO_A| = R - R_a = \frac{2aR^2}{bc + 2aR} = \frac{2aR}{bc} R_a.$$

Similarly, if O_B is the circumcenter of triangle BZ_3X_4, then

$$|OO_B| = R - R_b = \frac{2bR^2}{ca + 2bR} = \frac{2bR}{ca} R_b.$$

The distance between the two circumcenters O_A and O_B can be computed using the law of cosines, and the fact that $\angle O_A O O_B = 2\angle C$:

$$\begin{aligned}
|O_A O_B|^2 &= (R - R_a)^2 + (R - R_b)^2 - 2(R - R_a)(R - R_b)\cos 2C \\
&= (R - R_a)^2 + (R - R_b)^2 - 2(R - R_a)(R - R_b)(1 - 2\sin^2 C) \\
&= (R_a - R_b)^2 + 4(R - R_a)(R - R_b)\sin^2 C \\
&= (R_a - R_b)^2 + 4R_a R_b \cdot \frac{4R^2 \sin^2 C}{c^2} \\
&= (R_a - R_b)^2 + 4R_a R_b \\
&= (R_a + R_b)^2,
\end{aligned}$$

since $R = c/(2\sin C)$. This shows that the two circumcircles are tangent to each other externally. The same calculation shows that the other two pairs of circumcircles are also tangent externally.

Evidently, each of these Lucas circles is tangent internally to the circumcircle.

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How to Integrate A Polynomial Over A Sphere

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Several recent articles in the MONTHLY ([1], [2], [4]) have involved finding the area of n-dimensional balls or spheres or integrating polynomials over such sets. None of these articles, however, makes use of the most elegant and painless method for performing such calculations, which is to reverse the usual trick for computing $\int_{-\infty}^{\infty} e^{-x^2}\,dx$.

There is nothing new in this idea. It was shown to me in 1971 by V. Bargmann and E. Nelson, and I included it as an exercise in my book [3], whose first edition appeared in 1984. But the evidence suggests that it is not as universally known as it should be.

First, some notation: For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we set $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$, and we denote the sphere and ball of radius r about the origin in $\mathbb{R}^n$ by $S_n(r)$ and $B_n(r)$, respectively:

$$S_n(r) = \{x \in \mathbb{R}^n : |x| = r\}, \qquad B_n(r) = \{x \in \mathbb{R}^n : |x| < r\}.$$

We also set

$$S_n = S_n(1), \qquad B_n = B_n(1).$$

Any nonzero $x \in \mathbb{R}^n$ can be written uniquely in "polar coordinates":

$$x = rx', \quad \text{where} \quad r = |x| \quad \text{and} \quad x' = \frac{x}{|x|} \in S_n.$$

Let σ denote the $(n-1)$-dimensional surface measure on S_n; then the formula for integration in polar coordinates is

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_{S_n} \int_0^{\infty} f(rx')\, r^{n-1}\,dr\,d\sigma(x'). \tag{1}$$

(A brief derivation is sketched in Remark 2.) Our object is to compute $\int_{S_n} P\,d\sigma$ where P is a polynomial, and for this it suffices to consider the case where P is a monomial,

$$P(x) = x^{\alpha} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \qquad \big(\alpha_1, \ldots, \alpha_n \in \{0, 1, 2, \ldots\}\big). \tag{2}$$