Continued Fractions

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Abstract. In this paper, we discuss continued fractions. First, we discuss the definition and notation. Second, we discuss the development of the subject throughout history. Third, we recall some theory of continued fractions. Finally, we use the theory to examine applications of continued fractions.

1. Introduction. Continued fractions offer a useful means of expressing numbers and functions. In the early ages, 300 B.C.–200 A.D., mathematicians used other algorithms and methods to express numbers and to express solutions of Diophantine equations. Many of these algorithms were studied and modeled in the development of the continued fractions. Through the eighteenth century, use of continued fractions was limited to the area of mathematics.

Since the beginning of the twentieth century, continued fractions have become more common in various other areas. For example, Robert M. Corless's 1992 paper [**3**] examines the connection between chaos theory and continued fractions. They have also been used in computer algorithms for computing rational approximations to real numbers, as well as for solving Diophantine and Pell's equations. This paper provides an introduction to continued fractions. In Section 2, we discuss the general form of a continued fraction. We also give some definitions and notations that are needed in later sections. In Section 3, we trace the development of continued fractions over the past 2,500 years. In Section 4, we begin looking at the theory of continued fractions. In particular, we are going to see how to express real numbers as continued fractions. In Section 5, we continue our look at the theory. We are going to see how to express rational numbers as continued fractions. We note some similarities between this process and the Euclidean algorithm. In Section 6, we discuss the kth convergent of a continued fraction. Finally, in Section 7, we look at some applications. We are going to see how continued fractions can be used to find solutions of quadratic equations, to express irrational numbers, and to find factors of large numbers.

2. General Form. A *continued fraction* is an expression of the form

$$r =_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \dots}}}$$

where a_i and b_i are either rational numbers, real numbers, or complex numbers. If $b_i = 1$ for all i, then the expression is called a *simple* continued fraction. If the expression contains finitely many terms, then it is called a *finite* continued fraction; otherwise, it is called an *infinite* continued fraction. The numbers a_i are called the *partial quotients*.

If the expression is truncated after k partial quotients, then the value of the resulting expression is called the *kth convergent* of the continued fraction; it is denoted by C_k. If the a_i and b_i repeat cyclically, then the expression is called a *periodic* continued fraction. Due to the complexity of the expression above, mathematicians have adopted several more convenient notations for simple continued fractions. The most common of these is

$$r = [a_0, a_1, a_2, a_3, \ldots]\,.$$

If a_0 is an integer, then it is often separated from the rest of the coefficients with a semicolon:

$$r = [a_0; a_1, a_2, a_3, \ldots]. \tag{2-1}$$

Below, two other simpler notations are listed:

$$r = a_1 + \frac{1}{a_2+} + \frac{1}{a_3+} + \frac{1}{a_4+} + \ldots; \tag{2-2}$$

$$r = a_1 + 1(a_2 + 1/(a_3 + 1/(\ldots)). \tag{2-3}$$

In the remainder of this paper, we will use these simpler notations when possible.

3. Development. Early traces of continued fractions appear as far back as 306 B.C. Other records have been found that show that the Indian mathematician Aryabhata (475–550) used a continued fraction to solve a linear equation [**5**, pp. 28–32]. However, he did not develop a general method; rather, he used continued fractions only in specific examples. Continued fractions were used only in specific examples for more than 1,000 years. In the sixteenth century, two Italian mathematicians, Rafael Bombelli (1526–72) and Pietro Cataldi (1548–626), expressed $\sqrt{13}$ and $\sqrt{18}$, respectively, as periodic continued fractions. Both mathematicians only provided these examples; they stopped short of further investigation. John Wallis (1616–703) did go further, and through his work, continued fractions became a subject of study in its own right. First, in his 1656 book *Arithemetica Infinitorium*, he worked out the formula,

$$\frac{4}{\pi} = \frac{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times 9 \times \ldots}{2 \times 4 \times 4 \times 6 \times 6 \times 8 \times 9 \times \ldots}\,.$$

Although the right-hand side is not a continued fraction, Lord Brouncker (1620–84) rewrote it as follows:

$$\frac{4}{\pi} = 1 + \frac{1^2}{2+} + \frac{3^2}{2+} + \frac{7^2}{2+} + \ldots\,.$$

Brouncker did not go further with continued fractions. On the other hand, Wallis then took the first steps toward a general theory.

In his 1695 book, *Opera Mathematica*, Wallis explained how to compute convergents, and discovered some of their important properties. He also introduced the term "continued fraction." Earlier, they were known as "anthyphairetic ratios." The Dutch mathematician and astronomer, Christiaan Huygens (1629–95), made the first practical application of the theory in 1687. He wrote a paper explaining how to use convergents to find the best rational approximations for gear ratios. These approximations enabled him to pick the gears with the best numbers of teeth. His work was motivated by his desire to build a mechanical planetarium. Wallis and Huygens wrote the first general works on continued fractions. Later, the theory grew when Leonard Euler (1707–83), Johan Lambert (1728–77), and Joseph Louis Lagrange (1736–1813) worked on the topic. Much of the modern theory was developed in Euler's 1737 work, *De Fractionbous Continuis*. He showed that every rational number can be expressed as a finite simple continued fraction. He also gave the following expression for e as a continued fraction:

$$e - 1 = 1 + \frac{1}{1+} + \frac{1}{1+} + \frac{1}{2+} + \frac{1}{1+} + \frac{1}{1+} + \frac{1}{4+} + \frac{1}{1+} + \frac{1}{1+} + \frac{1}{6+} + \dots .$$

He used this expression to show that e and e^2 are irrational. He also showed how to go from a series to a continued fraction, and back. In 1763, Lambert generalized Euler's work on e to show that both e^x and $\tan x$ are irrational if x is rational. Lagrange used continued fractions to find the value of an irrational root of a quadratic equation. He also proved that a real irrational root is given by a periodic continued fraction. "The nineteenth century can be said to be a popular period for continued fractions," according to Claude Brezinski [**1**, p. 12]. It was a time in which "the subject was known to every mathematician." As a result, there was explosive growth, especially in the part concerning convergents (see Theorem 6-1 and Theorem 6-2 below). Also studied were continued fractions with complex numbers as terms. Among those to contribute were Karl Jacobi (1804–51), Oskar Perron (1880–1975), Charles Hermite (1822–1901), Karl Friedrich Gauss (1777–1855), Augustin Cauchy (1789–1857), and Thomas Stieltjes (1856–94), see [**2**, pp. 125–28]. By the beginning of the twentieth century, the theory had advanced greatly beyond the initial work of Wallis.

4. Expression of Real Numbers. Every real number x is represented by a point on the real line, and falls between two successive integers, say n and $n+1$:

$$n \le x < n + 1 .$$

In the case where x is an integer, then $n = x$. The integer n is often called the *floor* of x, and is written as

$$n = \lfloor x \rfloor .$$

The number $u = x - n$ satisfies $0 \le u < 1$. Thus, for a given real x there is a unique decomposition,

$$x = n + u ,$$

where n is an integer and u satisfies $0 \le u < 1$. Furthermore, $u = 0$ if and only if x is an integer. This decomposition is called the *mod one decomposition* of a real number. It is the first step in the process of expanding x as a continued fraction. The process of finding the continued fraction expansion of a real number is a recursive process. Given x, we begin with the mod one decomposition

$$x = n_1 + u_1, \tag{4-1}$$

where n_1 is an integer and $0 \le u < 1$.

If $u_1 = 0$, then the recursive process terminates with this first step. If $u_1 > 0$, then the reciprocal $1/u_1$ of u_1 satisfies $1/u_1 > 1$ since u_1 satisfies $0 \le u < 1$. In this case, the second step of the recursion is to apply the mod one decomposition to $1/u_1$, which yields

$$1/u_1 = n_2 + u_2, \tag{4-2}$$

where n_2 is an integer and u_2 satisfies $0 \leq u_2 < 1$. Combining (4-1) and (4-2), we see that

$$x = n_1 + \frac{1}{n_2 + u_2}\,.$$

In general, if $u_k = 0$, then the recursive process ends with

$$x = n_{k-1} + \frac{1}{n_k}\,.$$

In the case that $u_k > 0$, we can rewrite $1/u_k$ as

$$1/u_k = n_{k+1} + u_{k+1}\,,$$

where n_{k+1} is an integer u_{k+1} and satisfies $0 \leq u_{k+1} < 1$. After k steps, we can write the real number as

$$x = n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{\cdots + \cfrac{1}{n_k + u_k}}}}\,.$$

We can express any real number as a continued fraction using the above recursive process.

5. Expression of Rational Numbers. The process of expressing a rational number as a continued fraction is essentially identical to the process of applying the Euclidean algorithm to the pair of integers given by its numerator and denominator in lowest terms. Let $x = a/b$, with $b > 0$, be a representation of a rational number x. The mod one decomposition,

$$\frac{a}{b} = n_1 + u_1, \text{ where } u_1 = \frac{a - n_1 b}{b}\,,$$

shows that $u_1 = r_1/b$, where r_1 is the remainder on division of a by b. The case where $u_1 = 0$ is the case where x is an integer. Otherwise u_1 and the mod one decomposition of $1/u_1$ gives

$$\frac{b}{r_1} = n_2 + u_2, \text{ where } u_2 = \frac{b - n_2 r_1}{r_1}\,.$$

Hence $u_2 = r_2/r_1$ where r_2 is the remainder on division of b by r_1. The successive quotients in the Euclidean algorithm are the integers $n_1, n_2, \ldots$ occurring in the continued fraction. The Euclidean algorithm terminates after a finite number of steps with the appearance of a zero remainder. Therefore, the continued fraction of every rational number is finite. We can now prove the following theorem.

Theorem 5-1. *The continued fraction expression of a real number is finite if and only if the real number is rational.*

Proof: We have just shown that, if x is rational, then the continued fraction expansion of x is finite. To show the converse, we prove by induction that, if a simple continued fraction has n terms, then it is rational. Let X represent the value of the continued fraction. We first check the base case of $n = 1$. Then

$$X = a_1\,.$$

Since a_1 is an integer, X is rational. Thus, the base case is true. We now prove the inductive case. We assume that the statement is true for all $i \leq k$, and show that the statement is true for $k+1$. Let X be a continued fraction that is represented by $n+1$ terms. We want to show that X is rational. So, we have

$$X = [a_1; a_2, a_3, \ldots, a_n, a_{n+1}] .$$

We can rewrite this expression as

$$X = a_1 + \frac{1}{Y} ,$$

where $Y = [a_2; a_3, \ldots, a_n, a_{n+1}]$. The continued fraction X now has n terms, and by hypothesis it can be written as follows:

$$X = a_1 + \frac{1}{\frac{p}{q}} .$$

By doing some algebra, we see that

$$X = \frac{pa_1 + q}{p} .$$

Since a_1, p, and q are integers, X must be a rational. Therefore, the theorem is true for $n+1$, and so by induction, it must hold for all integers. The proof is now complete.

Corollary 5-2. *If a real number is irrational, then its continued fraction expression is infinite.*

Corollary 5-2 follows directly from Theorem 5-1.

6. Convergents. Below are two theorems involving the convergents of a continued fraction.

Theorem 6-1. *Given a continued fraction $[a_1; a_2, a_3, \ldots, a_{n-1}, a_n]$, a numerator p_i and a denominator q_i of the ith convergent C_i are given for all $i \geq 0$ by the recursive formulas,*

$$p_i = a_i p_{i-1} + p_{i-2} ,$$

$$q_i = a_i q_{i-1} + q_{i-2} ,$$

where $p_{-1} = 0$, $p_0 = 1$, $q_{-1} = 1$, and $q_0 = 1$.

Proof: We prove this assertion by using induction. We first check the two base cases:

$$C_1 = \frac{p_1}{q_1} = a_1 = \frac{a_1}{1} = \frac{a_1 \cdot 1 + 0}{a_1 \cdot 0 + 1} ;$$

$$C_2 = \frac{p_2}{q_2} = \frac{q_1 q_2 + 1}{q_2} = \frac{a_2 p_1 + p_0}{a_2 q_1 + q_0} .$$

Thus both base cases are true. We now assume that the assertion is true for all $i \leq k$, and show that it is true for $k+1$. By the recursive formula,

$$C_{k+1} = \frac{p_{k+1}}{q_{k+1}} = [a_1, a_2, \ldots, a_k, a_{k+1}] .$$

We rewrite this fraction as follows:

$$C_{k+1} = \frac{p_{k+1}}{q_{k+1}} = [a_1, a_2, a_3, \ldots, a_k, a'_{k+1}]$$

where $a'_{k+1} = a_k + 1/a_{k+1}$. The continued fraction now has k terms, and by hypothesis,

$$\begin{aligned}
C_{k+1} &= \frac{a'_{k+1}p_{k-1} + p_{k-2}}{a'_{k+1}q_{k-1} + q_{k-2}} \\
&= \frac{(a_k a'_{k+1} + 1)p_{k-1} + a'_{k+1}p_{k-2}}{(a_k a'_{k+1} + 1)q_{k-1} + a'_{k+1}q_{k-2}} \\
&= \frac{a_k a'_{k+1} + p_{k-1} + p_{k-1} + a'_{k+1}p_{k-2}}{a_k a'_{k+1} + q_{k-1} + q_{k-1} + a'_{k+1}q_{k-2}} \\
&= \frac{a'_{k+1}(a_k p_{k-1} + p_{k-2}) + p_{k-1}}{a'_{k+1}(a_k q_{k-1} + q_{k-2}) + q_{k-1}} \\
&= \frac{a'_{k+1}p_k + p_{k-1}}{a'_{k+1}q_k + q_{k-1}} \,.
\end{aligned}$$

The last step used the induction hypothesis for the substitution. Thus, the theorem is true for $k+1$, and by induction, must hold for all integers. The proof is now complete.

Theorem 6-2 (Fundamental Recurrence Relation). *Let p_i and q_i be the convergents. Then*

$$p_i q_{i-1} - p_{i-1} q_k = (-1)^i \quad \textit{for all } i \geq 0.$$

Proof: We will prove this statement by using induction. We first check the two base cases, $i = 1$ and $i = 2$:

$$\begin{aligned}
p_0 q_{-1} - p_{-1} q_0 &= 1(1) - 0(0) = 1 = (-1)^0 \,; \\
p_1 q_0 - p_0 q_1 &= (a_1 p_0 + p_{-1})(0) - 1(a_1 q_0 + q_1) \\
&= 0 - 1(0 + 1) \\
&= -1 \\
&= -(1)^1 \,.
\end{aligned}$$

Both cases are true. We now assume that the statement is true for all $i \leq k$, and show that the statement is true for $k+1$:

$$\begin{aligned}
p_{k+1}q_k - p_k q_{k+1} &= (a_{k+1}p_k + p_{k-1})q_k - p_k(a_{k+1}q_k + q_{k-1}) \\
&= a_{k+1}p_k q_k + p_{k-1}q_k - a_{k+1}p_k q_k - p_k q_{k-1} \\
&= p_{k-1}q_k - p_k q_{k-1} \\
&= -(p_k q_{k-1} - p_{k-1}q_k) \\
&= -(-1)^k \\
&= (-1)^{k+1} \,.
\end{aligned}$$

Since the assertion is true for $k+1$, the assertion is true for all integers, by induction. The proof is now complete.

7. Applications. In this section we consider some examples of the applications of continued fractions in mathematics. Several common methods are used to solve quadratic equations, including the quadratic formula, "normal" factoring, and graphing the equation. A less common method involves continued fractions, see [**6**, pp. 32–37]. This method is used less often because it requires more calculations, and in many cases it reveals only one of the two solutions. For example, consider the quadratic equation, $x^2 - 5x + 6 = 0$. Factoring the left-hand side, we find the roots are 2 and 3.

We now use continued fractions. We begin by "solving" $x^2 - 5x + 6 = 0$ for x:

$$x = 5 - \frac{6}{x}. \tag{7-1}$$

Next, we express x as an infinite continued fraction. To do so we substitute the expression $5 - \frac{6}{x}$ in for x:

$$x = 5 - \frac{6}{5-} + \frac{6}{x}. \tag{7-2}$$

We repeat the substitution again, putting $5 - \frac{6}{x}$ into (7-2) and getting

$$x = 5 - \frac{6}{5-} + \frac{6}{5-} + \frac{6}{x}. \tag{7-3}$$

Continuing this process, we produce an infinite continued fraction. Finally, we approximate one of the roots. When we set $x = 1$ in the right-hand side of (7-1), we get -1. Taking this value and plugging it into (7-2), we see that $x = 11$. Placing 11 in (7-3), we see the result is $x = 4.4545$. If we continue this process, then we produce an infinite sequence that approaches 3. If we change the initial value, then we produce a different infinite sequence that approaches 3. Therefore one solution of this equation is $x = 3$, and there is no way we can compute the second solution $x = 2$ using this method. The method can be generalized and applied to solve any quadratic equation.

Some Diophantine equations can be solved using continued fractions, including Pell's equation, linear Diophantine equations, and congruence equations [**7**, pp. 120–27]:

$$x^2 - Py^2 = 1;$$
$$ax + by = c;$$
$$ax = b \pmod{m}.$$

Section 5 explained a method used to express a rational number as a continued fraction. Using continued fractions also provides a way to express an irrational number and to approximate its value. This expression allows us to study certain interesting properties of an irrational number. For example, consider the irrational number $\sqrt{2}$. At first glance, it may seem difficult to express this number as a continued fraction since its decimal representation never ends. However, using steps similar to the ones taken to express a rational number as a continued fraction, we can express $\sqrt{2}$ as one also. In general, any irrational number can be expressed as a continued fraction. To express $\sqrt{2}$ as a continued fraction, we observe that $\sqrt{2} > 1$,

$$\sqrt{2} = 1 + 1/x. \tag{7-4}$$

Solving for some real number x, we find that

$$x = 1/(\sqrt{2} - 1)\,.$$

To get rid of the square root in the denominator, we multiply the numerator and denominator by $(\sqrt{2}+1)$, obtaining

$$x = (\sqrt{2} + 1)\,.$$

Using (7-4) and simplifying, we obtain the equation,

$$x = 2 + 1/x.$$

This equation now looks like equation (7-1), which we obtained from the quadratic equation, $x^2 - 5x + 6 = 0$. Continuing as in Section 5, we obtain the continued fraction representation of

$$\sqrt{2} = [1; 2, 2, 2, 2, 2, 2, \dots].$$

Other square roots and irrational numbers can be expressed similarly as a continued fraction. In 1975, M.A. Morrison and J. Brillhart developed the Continued Fraction Factorization Algorithm, which is a prime factorization algorithm that uses residues produced by the continued fraction of $\sqrt{mN}$. Here N is the number to be factored and m is chosen as small as possible so that mN is a square. The algorithm solves the equation $x^2 = y^2 (\bmod\ N)$ by finding an m for which $m^2 (\bmod\ N)$ has the smallest possible value. This method has a theoretical runtime of $O(e^{\sqrt{2 \log N \log\log N}})$, and was the fastest prime factorization algorithm in use until the development of the quadratic sieve method. This method was developed in 1981 by Carl Pomerance and has a theoretical runtime of $O(e^{\sqrt{\log N \log\log N}})$.

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