The Derivative

Suppose that y = f(x). Recall that the instantaneous rate of change when x = a is

$$m_a = \lim_{x \to a} m_{a,x} = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

It is common to make a change in the variables:
Let h = x-a.
Then x = a+h.
When x goes to a, then h goes to 0. Hence the instantaneous rate of change is also

$$m_a = \lim_{x \to a} m_{a,x} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

This is used so often that it has a special name and special notation.

The **derivative** of y = f(x) at x = a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

In general, the **derivative** of y = f(x) is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

Computation. So far the way it is computed is exactly the way we have done (but there will be still faster ways to be learned in the course).
Example. If $f(x) = 2x^2 - 4x + 1$, find f ' (3).
Solution

$$f'(3) = \lim_{h \to 0} \frac{f(3+h)-f(3)}{h}$$
$$= \lim_{h \to 0} \frac{2(3+h)^2 - 4(3+h) + 1 -[2(3)^2-4(3)+1]}{h}$$
$$= \lim_{h \to 0} \frac{2(9+6h+h^2) - 12-4h + 1 -7}{h}$$
$$= \lim_{h \to 0} \frac{18+12h+2h^2- 12-4h + 1 -7}{h}$$
$$= \lim_{h \to 0} \frac{8h+2h^2}{h}$$
$$= \lim_{h \to 0} (8+2h)$$
$$= 8$$

Meaning:
1. If y = f(x), then f'(a) is the instantaneous rate of change (with respect to x) when x = a.
2. If y = f(x), then f'(a) is the slope of the line tangent to the curve when x = a.

Example. If $P(t) = 4t^2 + 3t + 2$, find P'(1).
Solution.

$$P'(1) = \lim_{h \to 0} \frac{P(1+h) - P(1)}{h}$$

$$= \lim_{h\to 0} \frac{4(1+h)^2+3(1+h)+2 - 9}{h}$$

$$= \lim_{h\to 0} \frac{4(1+2h+h^2)+3(1+h)+2 - 9}{h}$$

$$= \lim_{h\to 0} \frac{4+8h+4h^2+3+3h+2 - 9}{h}$$

$$= \lim_{h\to 0} \frac{11h+4h^2}{h}$$

$$= \lim_{h\to 0} 11+4h$$

$$= 11$$

Example. If $P(t) = 3t^2$,
(i) find P'(t).
(ii) find the slope of the line tangent to $P = 3t^2$ where t = 7
Solution.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

To get P'(t) we replace f by P and a by t:

$$P'(t) = \lim_{h\to 0} \frac{P(t+h)-P(t)}{h}$$

$$= \lim_{h\to 0} \frac{3(t+h)^2-3t^2}{h}$$

$$= \lim_{h\to 0} \frac{3(t^2+2th+h^2)-3t^2}{h}$$

$$= \lim_{h\to 0} \frac{3t^2+6th+3h^2-3t^2}{h}$$

$$= \lim_{h\to 0} \frac{6th+3h^2}{h}$$

$$= \lim_{h\to 0} \underline{6t+3h}$$

$$= 6t$$

Thus P'(t) = 6t.
(ii) The slope of the tangent line is P'(7).
But P'(t) = 6t, so P'(7) = 6(7) = 42.

Example. If f(x) = 1/x, find f ' (x).
Solution.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1/(x+h) - 1/x}{h}$$

$$= \lim_{h \to 0} \frac{x/[x(x+h)] - (x+h)/[x(x+h)]}{h}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{hx(x+h)}$$

$$= \lim_{h \to 0} \frac{-\ h}{hx(x+h)}$$

$$= \lim_{h \to 0} \frac{-\ 1}{x(x+h)}$$

$$= -\frac{1}{x^2}$$

Other notations for the derivative are often used:
If y = f(x), then
derivative = f ' (x) = dy/dx = $D_x f(x)$.
Effectively they mean the same thing. When calculus was discovered, Newton (in England) used a notation similar to f ' (x), while Leibnitz in continental Europe used dy/dx. Now both are used equally often.

Example. If $y = x^2$, find dy/dx
Solution. dy/dx = lim (f(x+h)-f(x)) / h

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} (2x+h) = 2x$$

In fact, there are some even quicker ways to compute the derivative.

QUICK RULES FOR DIFFERENTIATION
Theorem. If n is a positive integer, then $D_x (x^n) = n\, x^{n-1}$

Example. Find $D_x (x^4)$
Answer $4\, x^{4-1} = 4\, x^3$

Example. If $f(x) = x^5$, find f '(x) and f '(2). Find the instantaneous rate of change when x = 1.
Solution. f '(x) = $5\, x^4$ and f '(2) = $5(2)^4 = 80$.
The instantaneous rate of change is f ' (1) = 5.

Theorem. If n is a positive integer and A is constant then
$D_x (Ax^n) = An\, x^{n-1}$

Example. If $f(x) = 5x^3$, then $f'(x) = 15x^2$

Example. If $f(x) = \pi x$, then $f'(x) = \pi$.

Theorem. If A and B are constants, then $D_x (Ax + B) = A$.

Example. $D_x(4x - 3) = 4$.

Example. If $f(x) = x\sqrt{2} + \pi$, then $f'(x) = \sqrt{2}$

Theorem. $D_x [f(x) + g(x)] = D_x (f(x)) + D(g(x))$
and $D_x [f(x) - g(x)] = D_x (f(x)) - D(g(x))$

Example. Find $D_x (3x^4 +6x^3)$
Answer. $3(4)x^3 + 6(3)x^2 = 12x^3 + 18x^2$

Example. If $f(x) = 2x^3 + 5x$, find $f'(2)$.
Solution By quick rules, $f'(x) = 6x^2 + 5$.
Hence $f'(2) = 6(2)^2 + 5 = 29$.

This same procedure works for more than one term as well:

Example. If $f(x) = 4x^5 - 7x^3 + 2x$, find $f'(x)$.
Answer. $f'(x) = 20x^4 - 21x^2 + 2$

Example. If $f(x) = 4x^4 - 3x^2 + x - 3$, find $f'(x)$ and $f'(1)$.
Answer. $f'(x) = 16x^3 - 6x + 1$
so $f'(1) = 11$

Summary: To differentiate any polynomial, change each nonconstant term $a_k x^k$ to $k a_k x^{k-1}$ and delete the constant term (if any).

Theorem. If A is constant then
$D_x (A f(x)) = A D_x(f(x))$.

Example. Suppose a population of bacteria in a culture with a special nutriment is given by $P(t) = 3t^5 - \pi t^3 + 7t^2 - 2t + \sqrt{2}$ where t is in hours and P is in mg.
(a) Find $P'(t)$
(b) Find $P'(1)$ exactly
(c) Find $P'(1)$ to 5 significant figures.
(d) Find the instantaneous rate of change of P when $t = 1$ to 5 significant figures.
Solutions:
(a) $P'(t) = 15t^4 - 3\pi t^2 + 14t - 2$
(b) $P'(1) = 15 - 3\pi + 14 - 2 = 27 - 3\pi$ mg/hour
(c) $P'(1) = 17.575$ mg/hour
(For this last, note that you can use the button for π on your calculator.)
(d) 17.575 mg/hour

Example. Suppose that the number of milligrams in a population of bacteria in a culture t hours after the start of the experiment is given by

$P(t) = 3.1\, t^3 + 4\, t^2 + 2\, t + 7.$

(a) Find P'(t).

(b) Find the slope of the line tangent to the curve P(t) when t = 0.5.

(c) Find the equation of the line tangent to the curve P(t) when t = 0.5.

(d) Is P(t) increasing or decreasing when t = 0.5?

Solution

(a) $P'(t) = 9.3\, t^2 + 8\, t + 2$

(b) slope = $P'(0.5) = 9.3\,(0.5)^2 + 8(0.5) + 2 = 8.325$ mg/hour

(c) When t = 0.5, $P = 3.1(0.5)^3 + 4\,(0.5)^2 + 2(0.5) + 2 = 4.3875$

Hence the line has one point (0.5, 4.3875) and slope 8.325

P = 4.3875 + 8.325(t-0.5)

P = 8.325 t +0.225

(d) increasing. The tangent line is moving upward.

Why are the quick rules true?

Theorem. If A and B are constants, then $D_x\,(Ax + B) = A$.

Proof. Here f(x) = Ax+ B

$$
\begin{aligned}
\text{So } f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{A(x+h)+B - (Ax+B)}{h} \\
&= \lim_{h\to 0} \frac{Ax+Ah+B - Ax-B}{h} \\
&= \lim_{h\to 0} \frac{Ah}{h} \\
&= \lim_{h\to 0} A \\
&= A
\end{aligned}
$$

Theorem. $D_x\,[\, f(x) + g(x)] = D_x\,(f(x)) + D(g(x))$

Proof.

$$
\begin{aligned}
D_x\,[\, f(x) + g(x)] \\
&= \lim_{h\to 0} \frac{f(x+h)+g(+h) - [f(x)+g(x)]}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)+g(+h) - f(x)-g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)+g(+h)-g(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(+h)-g(x)}{h} \right] \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(+h)-g(x)}{h}
\end{aligned}
$$

$= f'(x) + g'(x)$
$= D_x(f(x)) + D(g(x))$

Theorem. If A is constant then
$D_x(A f(x)) = A D_x(f(x))$.
Proof.

$$D_x(A f(x)) = \lim_{h\to 0} \frac{A f(x+h) - A f(x)}{h}$$

$$= \lim_{h\to 0} \frac{A [f(x+h) - f(x)]}{h}$$

$$= A D_x(f(x))$$

I'll indicate other properties by example.

$D_x(x^2) = 2x$
Proof. $D_x(x^2)$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h\to 0} (2x+h)$$

$$= 2x$$

$D_x(x^3) = 3x^2$
Proof. $D_x(x^3)$

$$= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h\to 0} (3x^2 + 3xh + h^2)$$

$$= 3x^2$$