6. Differentiating the exponential and logarithm functions

We wish to find and use derivatives for functions of the form $f(x) = a^x$, where a is a constant. By far the most convenient such function for this purpose is the exponential function with base the special number e.

Definition. The number e, which is approximately 2.7182818284590..., is the number such that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

The number e is called **Euler's number**, after the great mathematician Leonard Euler (1707-1783).

The major reason for the use of e is the following theorem, which says that e^x is its own derivative:

Theorem. $D_x [e^x] = e^x$.

Proof. For any function f(x), recall that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Hence

$$D_x [e^x] = \lim_{h \to 0} \frac{e^{(x+h)} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x e^h - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x (e^h - 1)}{h}$$

$$= e^x \ (1) \qquad \text{[by the definition of e]}$$

$$= e^x.$$

We often call the function $f(x) = e^x$ the **exponential function**, since it involves taking an exponent.

Example. Find $D_x [5 e^x]$
Answer: $5 e^x$

Derivatives get substantially more complicated when we remember the chain rule:

Recall that the **composition** $g \cdot f$ of the functions f and g is the function
$(g \cdot f)(x) = g(f(x))$.
This means, "do the function f to x, then do g to the result."

Chain Rule. Suppose f and g have derivatives. Then
$$(g \bullet f)'(x) = g'(f(x))\, f\,'(x).$$

This means that we can use the chain rule for the exponential function as well:

Theorem. (Exponential chain rule) $D_x [e^{f(x)}] = e^{f(x)} D_x[f(x)]$.
Proof. This is the chain rule where $g(y) = e^y$.

Example. Find $D_x [e^{6x}]$
Solution. $D_x [e^{6x}] = e^{6x} D_x [6x] = 6 e^{6x}$

Example. Find $D_x [e^{x^2}]$
Solution. $D_x [e^{x^2}] = e^{x^2} D_x [x^2] = 2x e^{x^2}$

Example. Find $D_x [5 e^{3x}]$
Solution. $D_x [5 e^{3x}] = 5 e^{3x} D_x [3x] = 15 e^{3x}$

Example. Find $D_x [2x e^{4x}]$
Solution. $D_x [2x e^{4x}] = (2x) D_x [e^{4x}] + e^{4x} D_x [2x]$
[by the Product Rule]
$= (2x) e^{4x} D_x[4x] + e^{4x}\, 2$
$= (2x) e^{4x} (4) + 2 e^{4x}$
$= 8x e^{4x} + 2 e^{4x}$

Example. If $f(t) = \dfrac{e^{4t}}{1 + e^{2t}}$

find f '(t).

Solution. Use the quotient rule
$$f\,'(t) = \frac{(1 + e^{2t}) D_t[e^{4t}] - e^{4t} D_t[1 + e^{2t}]}{(1 + e^{2t})^2}$$

$$= \frac{(1 + e^{2t})\, 4 e^{4t} - e^{4t}\, 2 e^{2t}}{(1 + e^{2t})^2}$$

$$= \frac{4 e^{4t} + 4 e^{2t} e^{4t} - 2 e^{4t} e^{2t}}{(1 + e^{2t})^2}$$

$$= \frac{4e^{4t} + 4e^{6t} - 2e^{6t}}{(1 + e^{2t})^2}$$

$$= \frac{4e^{4t} + 2e^{6t}}{(1 + e^{2t})^2}$$

Closely related to the exponential function is the logarithm function. Recall the **natural logarithm** of x, written ln(x), is the number such that

$e^{\ln(x)} = x.$

Thus ln(x) is the power to which e is raised to yield the number x. It makes sense only if x>0.

Theorem. $D_X [\ln(x)] = 1/x$ if $x > 0$.

Why is it true that
$D_X [\ln(x)] = 1/x$ if $x > 0$?
The argument is a clever one. Let $g(x) = e^{[\ln(x)]}$.
Then by the Exponential Chain Rule we have
$D_X[g(x)] = e^{[\ln(x)]} D_X [\ln(x)]$
But $g(x) = e^{[\ln(x)]} = x$, so $D_X[g(x)] = 1$.
Hence
$1= e^{[\ln(x)]} D_X [\ln(x)]$
$1= x D_X [\ln(x)]$
$D_X [\ln(x)] = 1/x$

Example. If g(x) = 3 ln(x) find g '(2).
Solution. $g'(x) = 3 D_X [\ln(x)] = 3/x$.
Hence g '(2) = 3/2 = 1.5.

Theorem. (Logarithm chain rule)

$$D_X [\ln (f(x))] = \frac{D_X [f(x)]}{f(x)} = \frac{f'(x)}{f(x)}$$

Proof. Write g(x) = ln(x). Then
$D_X [\ln (f(x))] = D_X [g (f(x))]$
$= g'(f(x)) f'(x)$ [by the Chain Rule]
$= [1 / f(x)] f'(x)$
$= f'(x) / f(x)$

Example. Find $D_X [\ln (x^2 + 1)]$
Solution.
$D_X [\ln (x^2 + 1)]$

$$= \frac{D_x [x^2 + 1]}{x^2 + 1}$$

$$= \frac{2x}{x^2 + 1}$$

Example. Find f '(x) if f(x) = 2x ln $(x^2 + 1)$
Solution.
f '(x) = $D_x [2 x \ln (x^2 + 1)]$
= $(2x) D_x [\ln (x^2 + 1)] + [\ln (x^2 + 1)] D_x [2 x]$

$$= \frac{(2x) D_x [x^2 + 1]}{x^2 + 1} + [\ln (x^2 + 1)] (2)$$

$$= \frac{(2x) (2x)}{x^2 + 1} + 2 \ln (x^2 + 1)$$

$$= \frac{4x^2}{x^2 + 1} + 2 \ln (x^2 + 1)$$

Example. Find f '(x) if $f(x) = \frac{\ln(x^2 + 1)}{x+1}$

Solution.
By the quotient rule we have

$$f'(x) = \frac{(x+1) D_x [\ln(x^2 + 1)] - [\ln(x^2 + 1)] D_x [x+1]}{(x+1)^2}$$

$$f'(x) = \frac{(x+1) [1/(x^2 + 1)] D_x [x^2 + 1] - [\ln(x^2 + 1)] (1)}{(x+1)^2}$$

$$f'(x) = \frac{(x+1) [1/(x^2 + 1)] (2x) - \ln(x^2 + 1)}{(x+1)^2}$$

$$f'(x) = \frac{(x+1)(2x) /(x^2 + 1) - \ln(x^2 + 1)}{(x+1)^2}$$

Simplify by multiplying numerator and denominator by (x^2+1).

$$f'(x) = \frac{(x+1)(2x) - (x^2 + 1)\ln(x^2 + 1)}{(x+1)^2 (x^2 + 1)}$$

Example. Find f '(x) if $f(x) = e^{3x} \ln(2x + 1)$.
Solution.

Use the product rule:

$f'(x) = e^{3x} D_x[\ln(2x + 1)] + \ln(2x + 1) D_x[e^{3x}]$

$f'(x) = e^{3x} [1/(2x + 1)]D_x[2x+1] + \ln(2x + 1) e^{3x} (3)$

$f'(x) = e^{3x} [2/(2x + 1)]+ 3 e^{3x} \ln(2x + 1)$

$f'(x) = 2 e^{3x} /(2x + 1)+ 3 e^{3x} \ln(2x + 1)$

We shall often have to solve equations involving exponents or logarithms. These will arise when we need to find critical numbers, or where a function is increasing of decreasing, or finding maxima or minima. These typically involve the identities

$\ln (e^c) = c$

$e^{\ln(b)} = b$

Example. Solve for x if $e^{2x} = 5$. Give your answer
(a) exactly (b) to 5 decimal places
Solution.

$e^{2x} = 5$

$\ln[e^{2x}] = \ln(5)$

$2 x = \ln(5)$

$x = [\ln(5)]/2$

Hence (a) has answer [ln(5)]/2.
To find the answer to (b), use your calculator: 0.804718956 to 5 decimal places means
0.80472

Example. Solve for x if $\ln(3x) - 4 = 0$. Give your answer
(a) exactly (b) to 5 decimal places
Solution

$\ln(3x) - 4 = 0$

$\ln(3x) = 4$

$e^{[\ln(3x)]} = e^4$

$3x = e^4$

(a) $x = (e^4)/3$

(b) $x = 18.19938$

Example. Solve for x if $e^{2x} - 5 x e^{2x} = 0$.
Solution.

$e^{2x} - 5 x e^{2x} = 0$

Factor:

$e^{2x} (1 - 5 x) = 0.$

If a product is 0, one of the factors is 0. Hence either

$e^{2x} = 0$ or $1-5x = 0$

But $e^{2x} > 0$ since it is a positive number raised to a power, so it is never 0. Hence

$1 - 5x = 0$

$5x = 1$
$x = 1/5$

Example. Solve for x if $x^2e^{3x} + 12\ e^{3x} = 7\ x\ e^{3x}$
Solution.
Rewrite as an expression = 0:
$x^2e^{3x} - 7\ x\ e^{3x} + 12\ e^{3x} = 0$
Factor
$e^{3x}\ (x^2 - 7\ x + 12\ ) = 0$
Since $e^{3x} > 0$, we must have
$x^2 - 7\ x + 12 = 0$
$(x-4)(x-3) = 0$
$x - 4 = 0$ or $x-3 = 0$
$x = 4$ or $x = 3$

Example. Solve for x if $3\ e^{-2x} - 5\ e^{-6x} = 0$. Give the answer to 4 decimal places.
Solution.
$3\ e^{-2x} - 5\ e^{-6x} = 0.$
$3\ e^{-2x} = 5\ e^{-6x}$
$3\ e^{-2x} /\ e^{-6x} = 5$
$3\ e^{-2x - (-6x)} = 5$
$3\ e^{4x} = 5$
$e^{4x} = 5/3$
$\ln e^{4x} = \ln(5/3)$
$4x = \ln(5/3)$
$x = (1/4) \ln(5/3) = 0.1277$

We can use these new functions to answer questions about relative extrema and absolute extrema.

Example. Let $f(x) = x\ e^{-2x}$.
(a) Tell where f is increasing, and where f is decreasing.
(b) Locate and classify all critical numbers.
(c) Find the absolute maximum and absolute minimum for $0 \le x \le 4$, and where they occur.

Solution.
$f\ '(x) = x\ D_x\ [e^{-2x}] + e^{-2x}\ D_x[x]$
$= x\ e^{-2x} D_x\ [-2x] + e^{-2x}\ (1)$
$= x\ e^{-2x}\ (-2) + e^{-2x}$
$= -2x\ e^{-2x} + e^{-2x}$

To find the critical numbers we solve $f\ '(x) = 0$:
$-2x\ e^{-2x} + e^{-2x} = 0$
Multiply by e^{2x}:
$-2x + 1 = 0$
$2x = 1$
$x = 1/2$

We now find the chart for $f\ '(x)$ by finding the value at points not equal to a critical number:

(+) 1/2 (-)

(a) f is increasing for x < 1/2.
f is decreasing for x > 1/2.

(b) The only critical number is
1/2 relative maximum

(c) For $0 \le x \le 4$ we form a chart including the endpoints and critical numbers in the interval

x	f(x)
1/2	0.1839
0	0
4	0.00134

The absolute maximum is 0.1839 at x = 1/2.
The absolute minimum is 0 at x = 0.

Example. Let $f(x) = \dfrac{\ln(x)}{2x}$ for x > 0.

(a) Tell where f is increasing, and where f is decreasing.
(b) Locate and classify all critical numbers.
(c) Find the absolute maximum and absolute minimum for $1 \le x \le 10$, and where they occur. Give answers to 3 decimal places

Solution. We must solve f '(x) = 0. By the quotient rule

$$f'(x) = \frac{2x\, D_x[\ln(x)] - (\ln(x))\, D_x[2x]}{(2x)^2}$$

$$f'(x) = \frac{2x\,[1/x] - (\ln(x))\,(2)}{4x^2}$$

$$f'(x) = \frac{2 - 2\ln(x)}{4x^2}$$

Hence f '(x) = 0 when

$2 - 2\ln(x) = 0$
$2\ln(x) = 2$
$\ln(x) = 1$
$e^{\ln(x)} = e^1$
$x = e$

We find the chart for f '(x) as follows, remembering that x>0. Note f(0) is meaningless.

0 (+) e (-)

Hence
(a) f is increasing for 0<x<e and f is decreasing for x>3.
(b) The only critical number is x = e, which is a relative maximum.

(c) We must consider the endpoints $x = 1$ and $x = 10$ together with critical numbers in the interval. Since $1<e<10$, we must consider $x= e$. Hence our chart of values is

x	f(x)
1	$\ln (1) / (2(1)) = 0$
10	$\ln (10) / 20 = 0.115$
e	$\ln (e) / (2e) = .184$

The absolute maximum is .184 at $x = e = 2.718$.
The absolute minimum is 0 at $x = 1$.

Example. The concentration $C(t)$ of a drug in the blood t hours after a pill is swallowed is given by $C(t) = 7 e^{-0.2t} - 7 e^{-0.5t}$ mg/l.

(a) When is the concentration at a maximum?
(b) What is the maximum concentration?
(c) For which t is the concentration increasing?
(d) For which t is the concentration decreasing?
Give answers to 2 decimal places.

Solution. We differentiate $C(t)$:

$C '(t) = 7 e^{-0.2t} (-0.2) - 7 e^{-0.5t} (-0.5)$
$= -1.4 e^{-0.2t} +3.5 e^{-0.5t}$
We need the critical numbers, so we solve $C '(t) = 0$.
$-1.4 e^{-0.2t} +3.5 e^{-0.5t} = 0$
Divide by $e^{-0.2t}$:
$-1.4 +3.5 e^{-0.5t} / e^{-0.2t} = 0$
$-1.4 +3.5 e^{-0.5t+0.2t} = 0$
$-1.4 +3.5 e^{-0.3t} = 0$
$3.5 e^{-0.3t} = 1.4$
$e^{-0.3t} = 1.4/3.5 = 0.4$
$\ln (e^{-0.3t}) = \ln(0.4)$
$-0.3 t = \ln (0.4)$
$t = (\ln (0.4))/(-0.3)$
$t = 3.05$ hours

Note the chart for C ' is then

(+)	3.05	(-)

Hence the answers are (a) $t = 3.05$ hours.
(b) $C(3.05) = 2.28$ mg/l
(c) $0<t< 3.05$ hours
(d) $t > 3.05$ hours