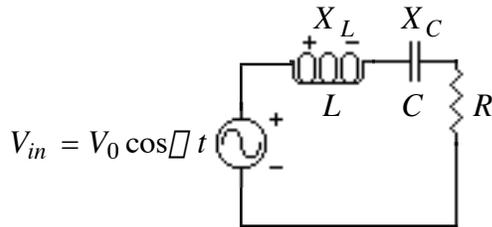


## R-L-C Circuits and Resonant Circuits

Consider the following RLC series circuit

•What's  $V_R$ ? Simplest way to solve for  $V$  is to use voltage divider equation in complex notation.



$$V_R = \frac{V_{in}R}{R + X_C + X_L}$$

$$= \frac{V_{in}R}{R + \frac{1}{j\omega C} + j\omega L}$$

Using complex notation for the apply voltage  $V_{in} = V_0 \cos \omega t = \text{Re al}(V_0 e^{j\omega t})$ ,

$$V_R = \frac{V_0 e^{j\omega t} R}{R + j\omega L + \frac{1}{\omega C}}$$

We are interested in the both the magnitude of  $V_R$  and its phase with respect to  $V_{in}$ .

First the magnitude:

$$|V_R| = \frac{|V_0 e^{j\omega t}| R}{\left| R + j\omega L + \frac{1}{\omega C} \right|}$$

$$= \frac{V_0 R}{\sqrt{R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2}}$$

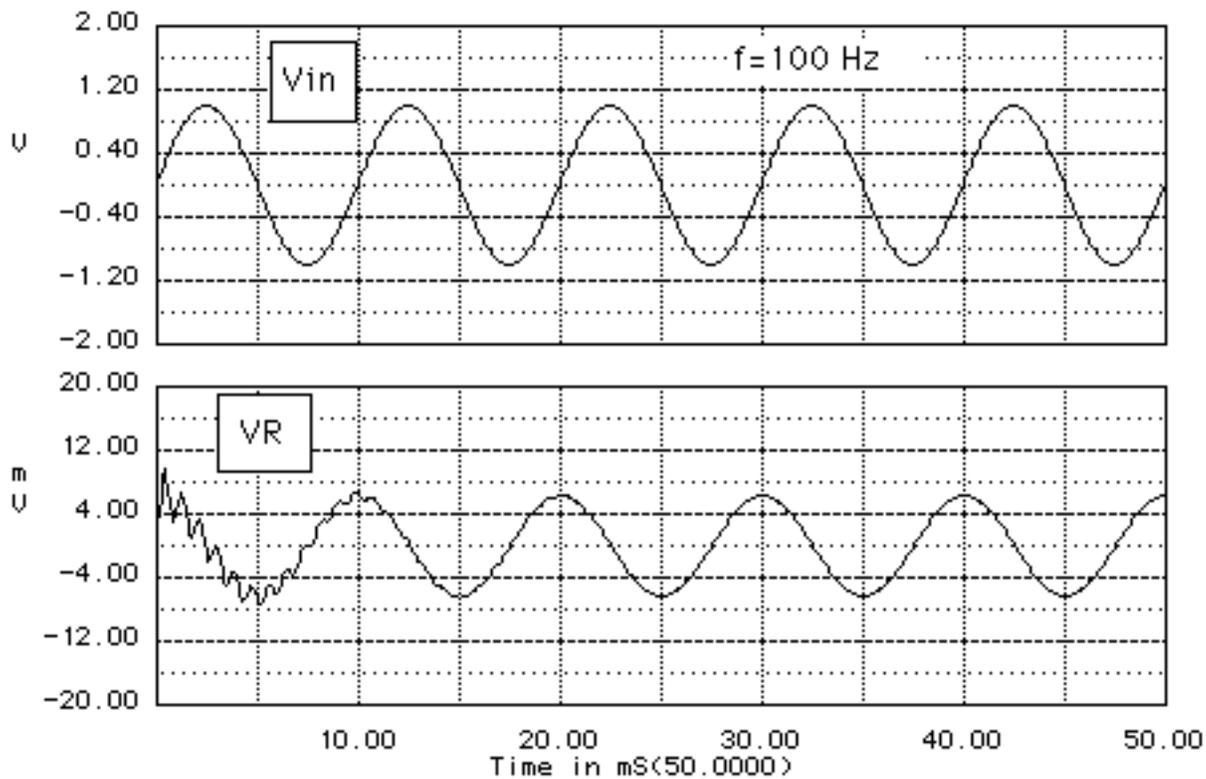
The following plots show  $V_R$  and  $V_{in}$  for an RLC circuit with:

$R = 100 \Omega$ ,  $L = 0.1 \text{ H}$ , and  $C = 0.1 \mu\text{F}$  at a frequency of 100 Hz.

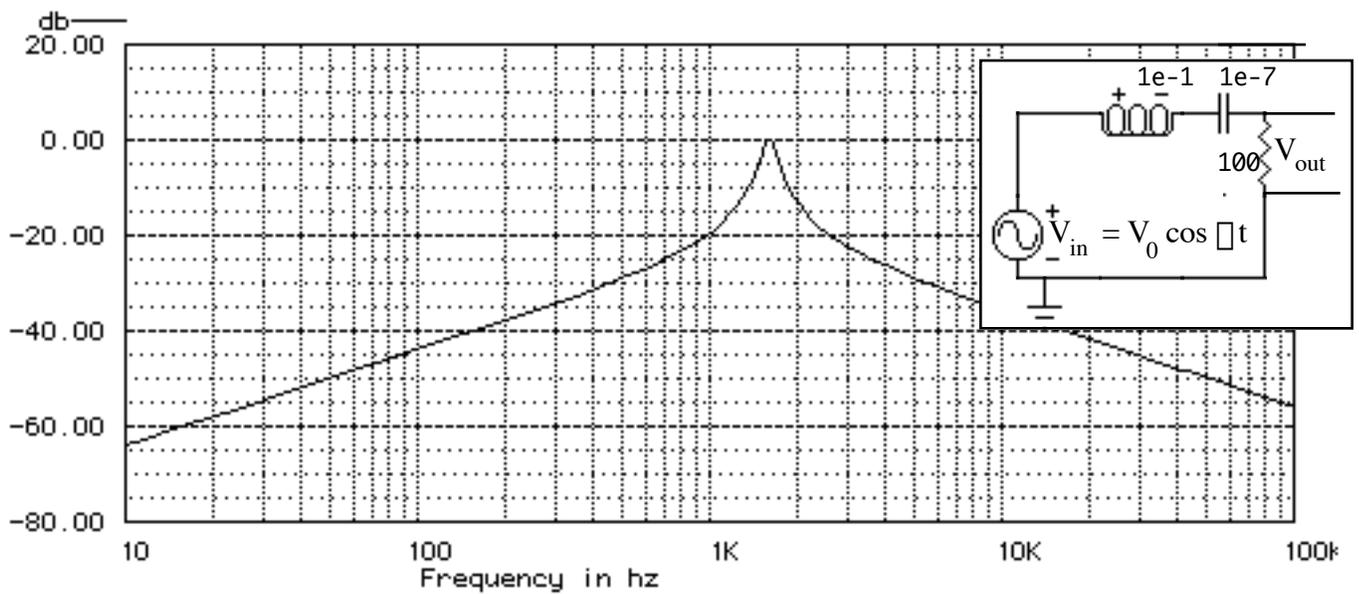
*Note:*  $V_R \ll V_{in}$  at this frequency.

$V_R$  and  $V_{in}$  are not in phase at this frequency.

The little wiggles on  $V_R$  are real! This behavior is due to the transient solution (homogeneous solution) to the differential eq. describing the circuit. After a few cycles this contribution to  $V_R$  has died out.



The following Bode plot shows the magnitude of  $V_R/V_{in}$  vs. frequency.



The *phase* of  $V_R$  with respect to  $V_{in}$  can be found by writing  $V_R$  in purely polar notation. For the denominator we have:

$$R + j\omega L \frac{1}{\omega C} = \sqrt{R^2 + \left(\omega L \frac{1}{\omega C}\right)^2} \exp\left\{j \tan^{-1} \frac{\omega L \frac{1}{\omega C}}{R}\right\}$$

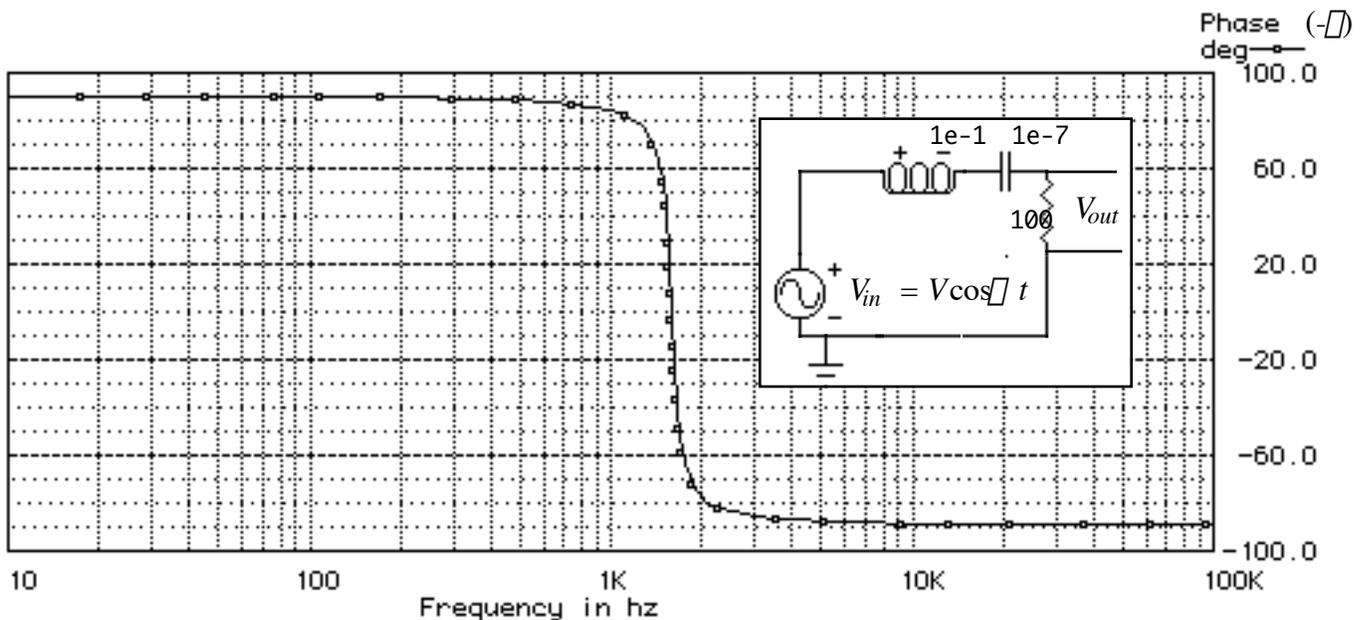
We can define the phase angle  $\phi$  using  $\tan \phi = \text{Imaginary } X / \text{Real } X$  for complex  $X$ . We can now write for  $V_R$  in complex form:

$$V_R = \frac{V_o R e^{j\omega t}}{e^{j\phi} \sqrt{R^2 + \left(\omega L \frac{1}{\omega C}\right)^2}} = |V_R| e^{j(\omega t - \phi)}$$

This phase angle is defined as:

$$\tan \phi = \frac{\omega L \frac{1}{\omega C}}{R}$$

Note: depending on  $L$ ,  $C$ , and  $\omega$ , the phase angle can be positive or negative! In this example, if  $\omega L > 1/\omega C$ , then  $V_R(t)$  **lags**  $V_{in}(t)$ .



Finally, we can write down the solution for  $V$  by taking the real part of the above equation:

$$V_R = \text{Re al} \frac{V_0 R e^{j(\omega t + \phi)}}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$$

$$= \frac{V_0 R \cos(\omega t + \phi)}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$$

•Some things to note:

In general  $V_C(t)$ ,  $V_R(t)$ , and  $V_L(t)$  are all out of phase with the applied voltage.

$I(t)$  and  $V_R(t)$  are in phase in a series RLC circuit.

The amplitude of  $V_C$ ,  $V_R$ , and  $V_L$  depend on  $\phi$ .

The table below summarizes the 3 cases with the following definitions:

$$Z = [R^2 + (\omega L - 1/\omega C)^2]^{1/2}$$

$$\tan \phi = (\omega L - 1/\omega C) / R$$

Voltage	Magnitude	Phase
$V_R$	$R/Z$	$-\phi$
$V_L$	$\omega L/Z$	$\phi/2 - \phi$
$V_C$	$1/\omega C Z$	$-\phi/2 - \phi$

•RLC circuits are resonant circuits, as the energy in the system "resonates" between the inductor and capacitor. "Ideal" capacitors and inductors do not dissipate energy. However, resistors dissipate energy or alternately, resistors do not store energy.

•Resonant Frequency: At the resonant frequency the imaginary part of the impedance vanishes.

For the series RLC circuit the impedance ( $Z$ ) is:

$$Z = R + X_L + X_C$$

$$= R + j(\omega L - 1/\omega C)$$

$$|Z| = [R^2 + (\omega L - 1/\omega C)^2]^{1/2}$$

At resonance (series, parallel etc), we have  $\omega L = 1/\omega C$  and:

$$\omega_R = \frac{1}{\sqrt{LC}}$$

At the resonant frequency the following are true for a series RLC circuit:

- $|V_R|$  is maximum (ideally =  $V_{in}$ )
- $\phi = 0$
- $\frac{|V_C|}{|V_{in}|} = \frac{|V_L|}{|V_{in}|} = \frac{\sqrt{L}}{R\sqrt{C}}$  ( $V_C$  or  $V_L$  can be  $> V_{in}$ !)

*The circuit acts like a narrow band pass filter.*

- There is an exact analogy between an RLC circuit and a harmonic oscillator (mass attached to spring):

$$m \frac{d^2x}{dt^2} + B \frac{dx}{dt} + kx = 0 \quad \text{damped harmonic oscillator}$$

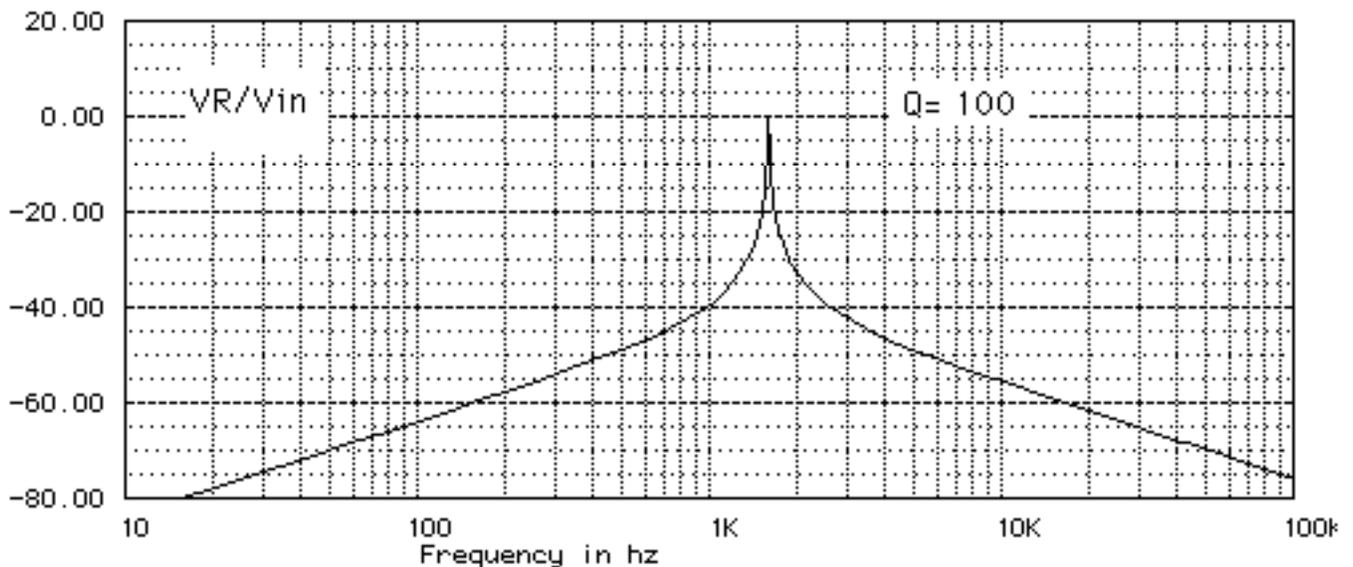
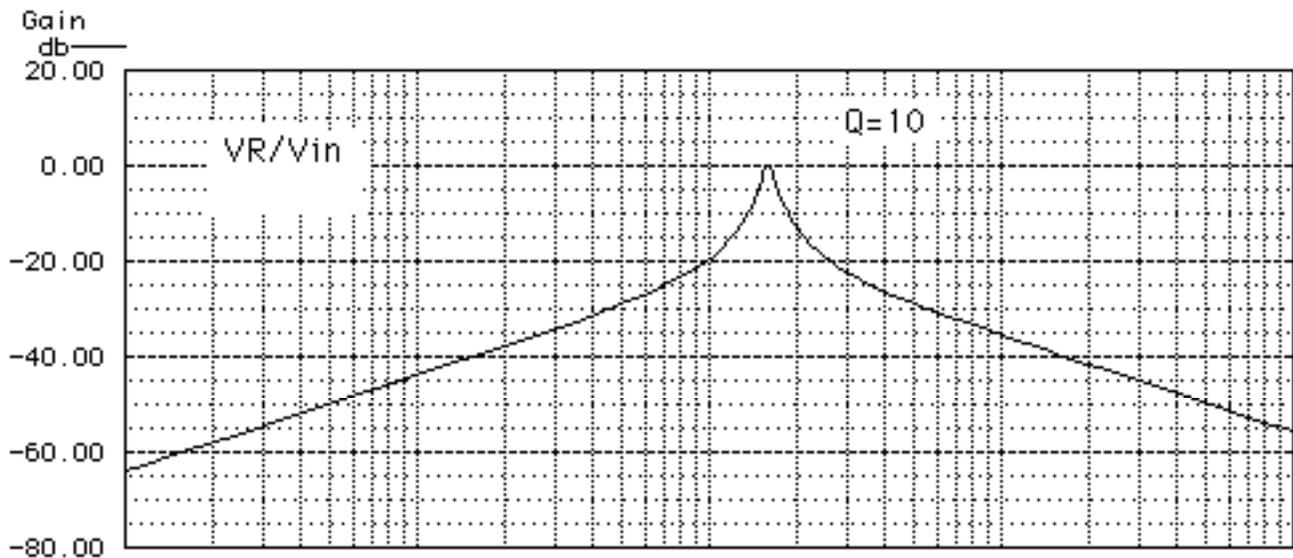
$$L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = 0 \quad \text{undriven RLC circuit}$$

$x \leftrightarrow q$  (electric charge),  $L \leftrightarrow m$ ,  $k \leftrightarrow 1/C$

$B$  (coefficient of damping)  $\leftrightarrow R$

- Q (quality factor) of a circuit: determines how well the RLC circuit stores energy  
 $Q = 2 \times (\text{max energy stored}) / (\text{energy lost})$  per cycle

Q is related to sharpness of the resonance peak:



The maximum energy stored in the inductor is  $LI^2/2$  with  $I = I_{MAX}$ . There is no energy stored in the capacitor at this instant because  $I$  and  $V_C$  are  $90^\circ$  out of phase.

The energy lost in one cycle is (Power)x(time for cycle) =  $I_{RMS}^2 R \frac{2\pi}{\omega_R} = \frac{1}{2} I_{max}^2 R \frac{2\pi}{\omega_R}$

$$Q = \frac{2\pi \frac{LI_{Max}^2}{2}}{\frac{2\pi}{\omega_R} \frac{RI_{Max}^2}{2}} = \frac{\omega_R L}{R}$$

There is another popular, equivalent expression for Q

$$Q = \frac{\omega_R}{\omega_U - \omega_L}$$

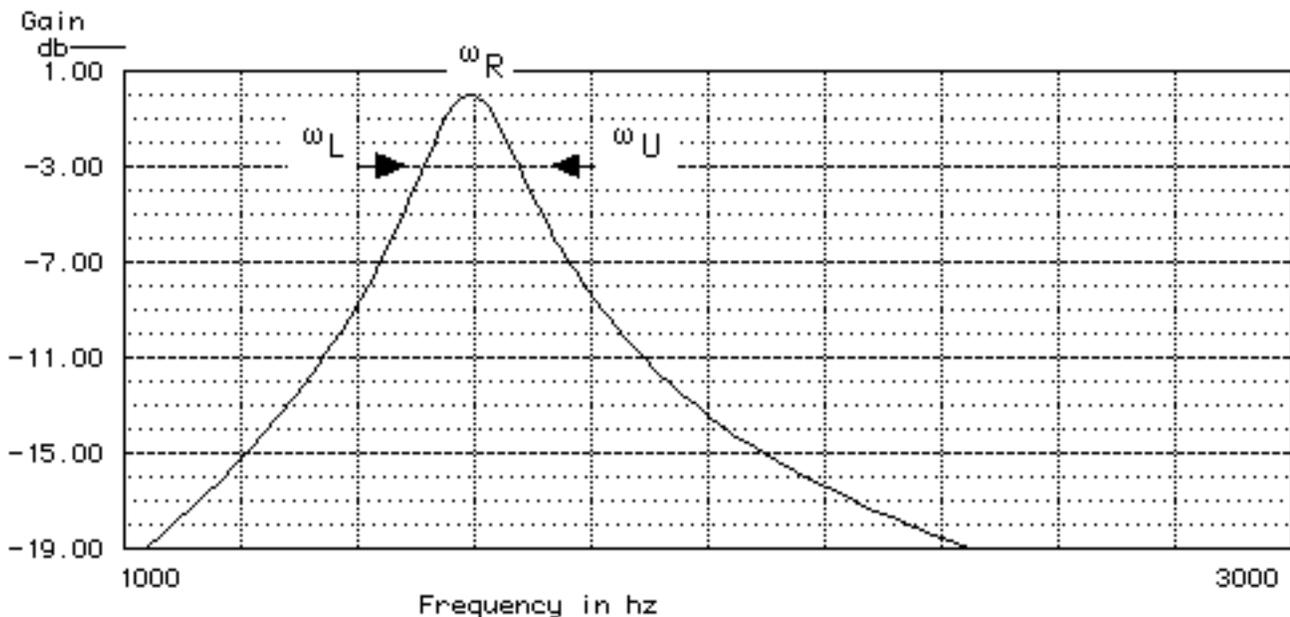
where  $\omega_U$  ( $\omega_L$ ) is the upper (lower) 3 dB frequency of the resonance curve. Q is related to sharpness of the resonance peak. I'll skip the derivation here as it involves a bit of algebra. However the two crucial points of the derivation include noting that:

$$\frac{V_R}{V_{in}} = \frac{1}{\sqrt{1 + Q^2 \left( \frac{\omega}{\omega_R} - \frac{\omega_R}{\omega} \right)^2}}$$

and at the upper and lower 3 dB points:

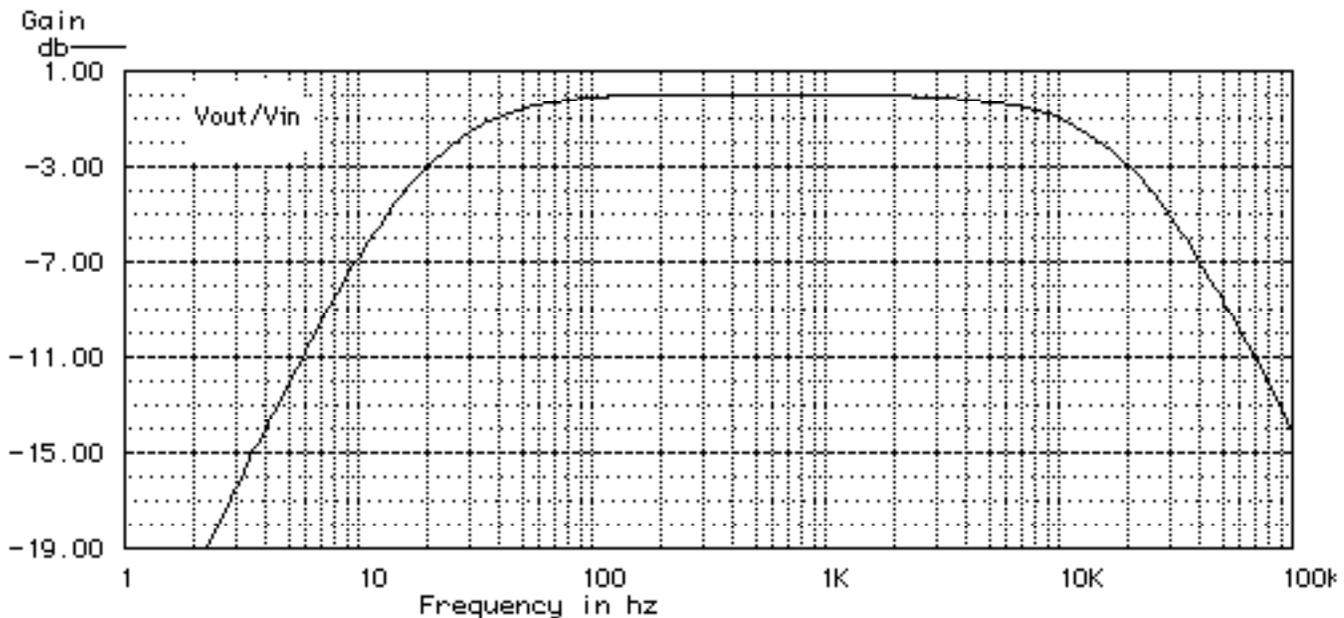
$$Q \left( \frac{\omega}{\omega_R} - \frac{\omega_R}{\omega} \right) = \pm 1$$

Note: Q can be measured from the shape of the resonance curve, one does not need to know R, L, or C to find Q!

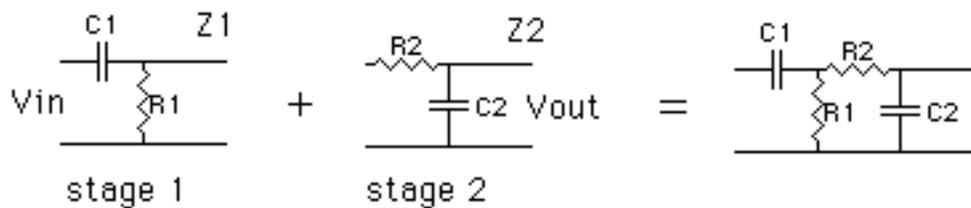


Example: Audio filter (band pass filter)

Audio filter is matched to the frequency range of the ear (20-20,000 Hz).



Let's design an audio filter using low and high pass RC circuits.



Ideally, the frequency response is flat over 20-20,000 Hz, and rolls off sharply at frequencies below 20 Hz and above 20,000 Hz. Set 3 dB points as follows:

$$\text{lower 3 dB point : } 20 \text{ Hz} = 1/2\sqrt{R_1 C_1}$$

$$\text{upper 3 dB point: } 2 \times 10^4 \text{ Hz} = 1/2\sqrt{R_2 C_2}$$

If we put these two filters together we don't want the 2<sup>nd</sup> stage to affect the 1<sup>st</sup> stage. We can accomplish this by making the impedance of the 2<sup>nd</sup> ( $Z_2$ ) stage much larger than  $R_1$ . Remember  $R_1$  is in parallel with  $Z_2$ .

$$Z_1 = R_1 + 1/j\omega C_1$$

$$Z_2 = R_2 + 1/j\omega C_2$$

In order to insure that the second stage does not "load" down the first stage we need:

$$R_2 \gg R_1 \text{ since at high frequencies } Z_2 \approx R_2$$

We can now pick and calculate values for the  $R$ 's and  $C$ 's in the problem.

$$\text{Let } C_1 = 1 \mu\text{F, then } R_1 = 1/(20\text{Hz } 2\sqrt{C_1}) = 8 \text{ k}\Omega$$

$$\text{Let } R_2 > 100R_1 \approx R_2 = 1 \text{ M}\Omega, \text{ and } C_2 = 1/(2 \times 10^4 \text{ Hz } 2\sqrt{R_2}) = 8 \text{ pf}$$

Thus we find the following  $R$ 's and  $C$ 's:

$$R_1 = 8 \text{ k}\Omega, C_1 = 1 \mu\text{F}$$

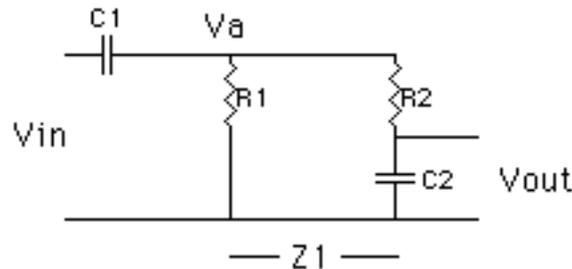
$$R_2 = 1 \text{ M}\Omega, C_2 = 8 \text{ pf}$$

•Exact derivation for above filter:

In the above circuit we treated the two RC filters as independent.

Why did this work?

We want to calculate the gain ( $V_{out}/V_{in}$ ) of the following circuit:



Working from right to left, we have:

$$V_{out} = V_a X_2 / (X_2 + R_2)$$

$$V_a = V_{in} Z_1 / Z_T$$

$Z_T$  is the total impedance of the circuit as seen from the input while  $Z_1$  is the parallel impedance of  $R_1$  and  $R_2$ , in series with  $C_2$ .

$$Z_1 = \frac{R_1(R_2 + X_2)}{R_1 + R_2 + X_2} \quad \text{and} \quad Z_T = X_1 + Z_1$$

We can now solve for  $V_a$ :

$$V_a = \frac{V_{in} R_1 (R_2 + X_2)}{X_1 (R_1 + R_2 + X_2) + R_1 (R_2 + X_2)}$$

Finally we can solve for the gain  $G = |V_{out}/V_{in}|$ :

$$\frac{V_{out}}{V_{in}} = \frac{R_1 X_2}{X_1 (R_1 + R_2 + X_2) + R_1 (R_2 + X_2)}$$

We can relate this to our previous result by rewriting the above as:

$$\frac{V_{out}}{V_{in}} = \frac{R_1 \frac{X_2}{R_2 + X_2}}{X_1 \left[ \frac{R_1}{R_2 + X_2} + 1 \right] + R_1}$$

If we now remember the approximation ( $R_1 \ll R_2 + X_2$ ) made on the previous page to insure that the second stage did not load down the first then we get the following:

$$\frac{V_{out}}{V_{in}} = \frac{R_1}{R_1 + X_1} \frac{X_2}{R_2 + X_2}$$

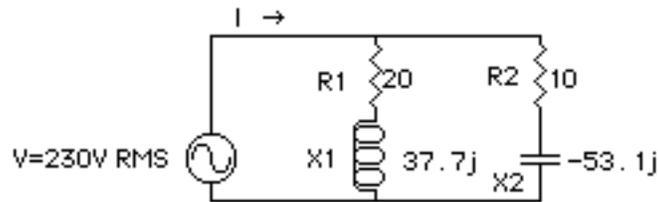
The gain of the circuit looks like the *product* of two filters, one high pass and one low pass!

If we calculate the gain of this circuit in dB, then the over all gain is the sum of the gain of each piece:

$$\begin{aligned} \text{Gain in dB} &= 20 \log \left[ \frac{V_{out}}{V_{in}} \right] \\ &= 20 \log \left[ \frac{R_1}{R_1 + X_1} \right] + 20 \log \left[ \frac{X_2}{R_2 + X_2} \right] \end{aligned}$$

The gain of successive filters measured in dB's add.

Another Example: Calculate  $|I|$  and the phase angle between  $V_{in}$  and  $I$  for the following circuit:



a) First calculate  $|I|$ .

The total current out of the input source ( $I$ ) is related to  $V_{in}$  and the total impedance ( $Z_T$ ) of the circuit by Ohm's law:

$$I = V_{in} / Z_T$$

The total impedance of the circuit is given by the parallel impedance of the two branches:

$$1 / Z_T = 1 / Z_1 + 1 / Z_2$$

$$Z_1 = R_1 + X_1$$

$$Z_2 = R_2 + X_2$$

Putting in numerical values for the  $R$ 's and  $X$ 's we have:

$$Z_1 = 20 + j37.7 \Omega$$

$$Z_2 = 10 - j53.1 \Omega$$

$$Z_T = 67.4 + j11.8 \Omega$$

We can now find the magnitude of the current (an RMS value since  $|V_{in}|$  is given as RMS).

$$\begin{aligned} |I| &= |V_{in}| / |Z_T| \\ &= 230 \text{ V} / 68.4 \Omega \\ &= 3.36 \text{ A} \end{aligned}$$

b) Calculate the phase angle between  $V_{in}$  and  $I$ :

Its easiest to solve this by writing  $V$  and  $Z$  in polar form:

$$V_{in} = (230)e^{j\omega t}$$

$$Z_T = (68.4)e^{j\phi}$$

$$\begin{aligned} \tan \phi &= \text{Im } Z_T / \text{Re } Z_T \\ &= 11.8 / 67.4 \end{aligned}$$

$$\phi = 9.9^\circ$$

Finally we can write for the current:

$$I = (230 / 68.4)e^{j(\omega t - \phi)}$$

Taking the real part of  $I$  we get:

$$I = 3.36 \cos(\omega t - 9.9^\circ) \text{ A}$$

Thus the current lags the voltage by  $9.9^\circ$ .