MAT A26
Lecture 40

1 Series and Their Sums

Definition 1. 1. *A real (or complex) series is an expression of the form*

$$\sum_{k=0}^{\infty} a_k$$

where a_k is a sequence of real (or complex) numbers.

2. *The nth partial sum of the series is*

$$s_n = \sum_{k=0}^{n} a_k$$

(Note that $s_0, s_1, s_2, s_3, \ldots$ is a sequence.)

3. *We say the series converges to L and write $\sum_{k=0}^{\infty} a_k = L$ if*

$$\lim_{n\to\infty} s_n = L$$

(We say that L is the sum of the series. If the limit doesn't exist we say the series diverges or doesn't converge.)

Example 1. *(Geometric Series) Suppose that $a \neq 0$. Show that*

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \qquad \text{if } |r| < 1$$

and that if $|r| \geq 1$ then the series doesn't converge.

SOLUTION: Consider the partial sum $s_n = \sum_{k=0}^{n} ar^k$.
If $r = 1$ then $s_n = (n+1)a$ so $\lim_{n\to\infty} s_n$ doesn't exist. So we may assume that $r \neq 1$.
Write

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} + ar^n \\ rs_n &= ar + ar^2 + \cdots + ar^n + ar^{n+1} \\ s_n - rs_n &= a - ar^{n+1} \\ (1-r)s_n &= a(1 - r^{n+1}) \end{aligned}$$

$$s_n = a\frac{1-r^{n+1}}{1-r} \qquad \text{since } r \neq 1$$

so

$$\lim_{n\to\infty} s_n = a\,\frac{1}{1-r} + a\,\frac{1}{1-r}\lim_{n\to\infty} r^{n+1}$$

Since $\lim_{n\to\infty} r^{n+1} = 0$ if $|r| < 1$ and doesn't exist if either $|r| > 1$ or $r = -1$, we get the result.

Example 2. *Sum the series* $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(1+2i)^k}$.

SOLUTION: This is a geometric series with $a = 1$ and $r = \dfrac{1}{1+2i}$. Moreover

$$|r| = \frac{1}{|1+2i|} = \frac{1}{\sqrt{1^2+2^2}} = \frac{1}{\sqrt{5}} < 1$$

Thus

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} = \frac{1}{1-\left(\frac{1}{1+2i}\right)} = \frac{1+2i}{1+2i-1} = \frac{1+2i}{2i} = 1 + \frac{1}{2i} = 1 - \frac{1}{2}i$$

Example 3. *Find the sum of* $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots$ *where the general term is* $a_k = \frac{1}{k(k+1)}$, *where* $k = 1, 2, \cdots$.

SOLUTION: Using the method of partial fractions, we find that

$$a_k = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

Thus

$$s_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

We see there is an immense cancellation. (When such a cancellation occurs we say that the series is a *telescoping series*, since it "folds up like a telescope".) This leaves us with the formula

$$s_n = 1 - \frac{1}{n+1} \qquad \text{so} \qquad \lim_{n\to\infty} s_n = 1$$

and so we have

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots = 1$$

Lemma 1.

$$\sum_{k=1}^{\infty} a_k \quad \textit{converges} \qquad \Longrightarrow \qquad \lim k \to \infty a_k = 0$$

PROOF: Let $s_n = \sum_{k=1}^n a_k$. Since the series converges, to L say, we have $\lim_{n\to\infty} s_n = L$. Now $a_k = s_k - s_{k-1}$, so

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} s_k - \lim_{k\to\infty} s_{k-1} = L - L = 0$$

Corollary 1. *If* $\lim_{k\to\infty} a_k \neq 0$ *then* $\sum_{k=1}^\infty a_k$ *diverges.*

REMARK: The corollary is more important than the theorem since it gives a simple test for showing that a series diverges.
However, be careful:
IT IS NOT TRUE THAT $\lim_{k\to\infty} a_k = 0$ **implies** $\sum_{k=1}^\infty a_k$ **converges.**

Example 4. *Does the series* $\displaystyle\sum_{k=0}^{\infty} \frac{k}{k+2}$ *converge?*

SOLUTION: As a first step, calculate $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k}{k+2} = 1$. Since this is not zero, the series diverges. (If we had gotten zero, WE WOULD NOT HAVE KNOWN if the series converged or diverged. The test would be inconclusive in this case, and we would have had to go on to further steps.)

Example 5. *(Harmonic Series) Does the series* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ *converge?*

SOLUTION; Group the terms in the series as follows:

$$1 + \left(\frac{1}{2}\right) + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> \frac{1}{2},\ \text{(4 terms, each} \geq \frac{1}{8})} + \underbrace{\left(\frac{1}{9} + \cdots + \frac{1}{16}\right)}_{> \frac{1}{2}\ \text{(8 terms, each} \geq \frac{1}{16})} + \cdots$$

The inequalities are seen by replacing each term in a group by the last term in the group. From this we see that $s_{2^n} \geq 1 + \frac{1}{2}n$ so the series **diverges**.

Perhaps you wouldn't have been able to see, by yourself, how to deal with this example. However, now we have dealt with it, it becomes a standard model against which many other series may be compared.

REMARK Note that, for the harmonic series, the individual terms do go to zero; nonetheless the series diverges. On the other hand the divergence takes place very slowly. We have

$$1 + \left(\frac{1}{2}\right) + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{< 1} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{< 1} + \underbrace{\left(\frac{1}{9} + \cdots + \frac{1}{16}\right)}_{< 1} + \cdots$$

so $s_{2^n} < 1 + n$. In particular, in order to have $s_k > N$ we would have to take k with more than $N \log_{10} e$ digits. For example to have $s_k > 1000$ we would need to take $k > 10^{434}$.

Example 6. *Show that the partial sums s_n for the harmonic series approximate the natural log of n in the sense that $\lim_{n\to\infty}(s_n - \ln n)$ exists.*

SOLUTION: To see convergence, note that

$$\begin{aligned}
t_n &\stackrel{\text{def}}{=} s_n - \ln(n+1) \\
&= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \int_1^{n+1} \frac{dt}{t} \\
&= \underbrace{\int_1^2 \left(1 - \frac{1}{t}\right) dt}_{>0} + \underbrace{\int_2^3 \left(\frac{1}{2} - \frac{1}{t}\right) dt}_{>0} + \cdots + \underbrace{\int_n^{n+1} \left(\frac{1}{n-1} - \frac{1}{t}\right) dt}_{>0} \\
&\le (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{n-1} - \frac{1}{n}) = 1 - \frac{1}{n} < 1
\end{aligned}$$

is monotone (increasing) and bounded, so by BMCT, it converges to some number $\gamma \le 1$. (It may be shown that $\gamma = 0.5772156649\cdots$.)