Lecture 6

Transmission Line II and Matching

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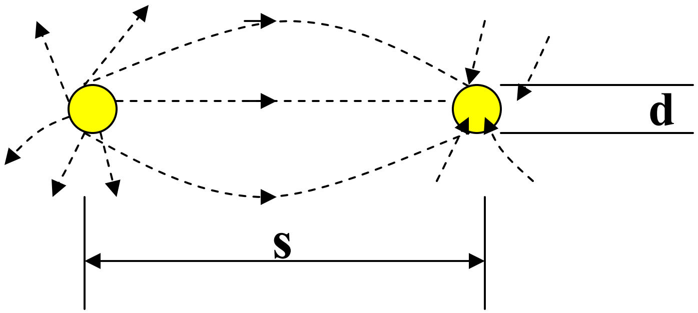
Types of Transmission Lines:

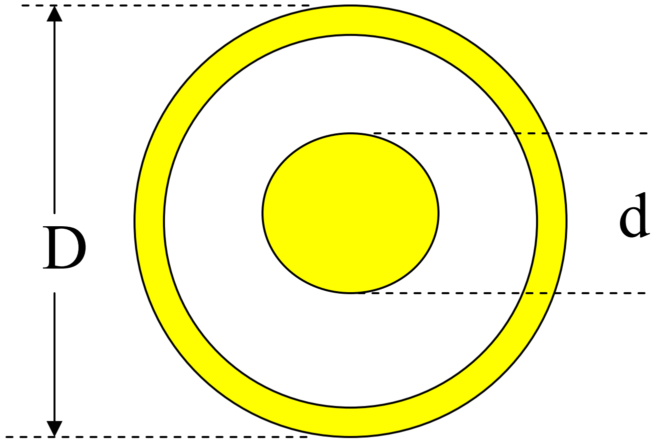


$$Z_\circ = \frac{120}{\sqrt{\varepsilon_r}} ln\frac{2s}{d}[\Omega]$$

$$L = \frac{\mu}{\pi} ln\frac{2s}{d}[H/m]$$

$$C = \frac{\pi\varepsilon}{ln\frac{2s}{d}}[F/m]$$

$$v = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\varepsilon}}[m/s]$$



$$Z_\circ = \frac{60}{\sqrt{\varepsilon_r}} ln\frac{D}{d}[\Omega]$$

$$L = \frac{\mu}{2\pi} ln\frac{D}{d}[H/m]$$

$$C = \frac{2\pi\varepsilon}{ln\frac{D}{d}}[F/m]$$

$$v = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\varepsilon}}[m/s]$$

- Transmission Lines are "Guiding" devices for carrying electromagnetic waves to and from antenna, etc.

- Transmission Line behavior occurs when the wavelength of the wave is small relative to the length of the cable.

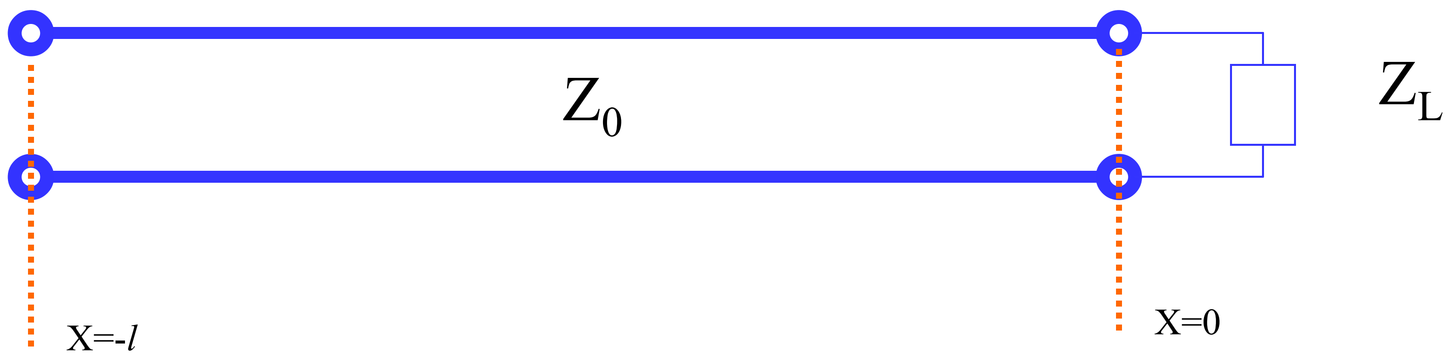
- We have already shown that in a loss-less line (zero resistance along conductors, infinite resistance between them), the propagation speed is given by

$$\nu = \frac{1}{\sqrt{LC}}[m/s]$$

where L and C are the inductance and capacitance per unit length. The characteristic (specific) impedance is given by:

$$Z_\circ = \sqrt{\frac{L}{C}}[\Omega]$$

Transmission Lines with Unmatched Termination:



Incident Wave: $\overline{v}_i e^{j(\omega t-\beta x)}$

Reflected Wave: $\overline{v}_r e^{j(\omega t+\beta x)}$

$$\therefore v(x,t)=\overline{v}_i e^{j(\omega t-\beta x)}+\overline{v}_r e^{j(\omega t+\beta x)}=\tilde{v}(x)e^{j\omega t} \quad \textit{where } \tilde{v}(x)=\overline{v}_i e^{j(-\beta x)}+\overline{v}_r e^{j(\beta x)}$$

Similarly: $$i(x,t)=\bar{i}(x)e^{j\omega t} \quad \textit{where } \bar{i}(x)=\frac{\overline{v}_i}{Z_\circ}e^{j(-\beta x)}-\frac{\overline{v}_r}{Z_\circ}e^{j(\beta x)}$$

Notice minus sign in reflected current component: Energy flows in opposite direction to incident wave.

Transmission Lines with Unmatched Termination:

Now $\dfrac{\tilde{v}(0)}{\tilde{i}(0)} = Z_L$ *and*

$$\left.\begin{aligned} \tilde{v}(0) &= \overline{v}_i + \overline{v}_r \\ \tilde{i}(0) &= \frac{\overline{v}_i}{Z_\circ} - \frac{\overline{v}_r}{Z_\circ} \end{aligned}\right\} Z_L = \frac{\overline{v}_i + \overline{v}_r}{\overline{v}_i - \overline{v}_r} \cdot Z_\circ$$

$$\therefore Z_L \overline{v}_i - Z_L \overline{v}_r = Z_\circ \overline{v}_i + Z_\circ \overline{v}_r$$

$$(Z_L - Z_\circ)\overline{v}_i = (Z_L + Z_\circ)\overline{v}_r$$

$$\therefore \frac{\overline{v}_r}{\overline{v}_i} = \textit{Reflected Coefficient} \Rightarrow \Gamma = \frac{Z_L - Z_\circ}{Z_L + Z_\circ}$$

Transmission Lines with Unmatched Termination:

$$\tilde{v}(x) = \bar{\mathrm{v}}_i \left(e^{-j\beta x} + \Gamma e^{j\beta x} \right)$$

$$\tilde{i}(x) = \frac{\bar{\mathrm{v}}_i}{Z_\circ} \left(e^{-j\beta x} - \Gamma e^{j\beta x} \right)$$

Generalized Impedance : $Z(x) = \dfrac{\tilde{v}(x)}{\tilde{i}(x)} = Z_\circ \dfrac{e^{-j\beta x} + \Gamma e^{j\beta x}}{e^{-j\beta x} - \Gamma e^{j\beta x}}$

$$= Z_\circ \frac{e^{-j\beta x} + \dfrac{Z_L - Z_\circ}{Z_L + Z_\circ} e^{j\beta x}}{e^{-j\beta x} - \dfrac{Z_L - Z_\circ}{Z_L + Z_\circ} e^{j\beta x}}$$

Transmission Lines with Unmatched Termination:

$$Suppose\ \ell \to \lambda/4\ [Quarter-Wavelength]$$

$$so\ \beta\ell \to \beta\frac{\lambda}{4} = \frac{2\pi}{\lambda}\times\frac{\lambda}{4} = \frac{\pi}{2} \quad And\ \tan\frac{\pi}{2} = \infty$$

$$\therefore Z\left(-\frac{\lambda}{4}\right) = \lim_{\ell\to\lambda/4}\left[Z(-\ell)\right] = Z_\circ\frac{jZ_\circ\tan\beta\ell}{jZ_L\tan\beta\ell} = \frac{Z_\circ^2}{Z_L}$$

$\therefore$ *We can match a line of impedance* Z_s *to a load of impedance* Z_L *with a* $\frac{\lambda}{4}$ *section of characteristic impedance* $Z_\circ$ *such that*

$$Z_\circ = \sqrt{Z_s Z_L}\ \Omega$$

Transmission Lines with Unmatched Termination:

$$= Z_\circ \left[\frac{(Z_L + Z_\circ)(\cos\beta x - j\sin\beta x) + (Z_L - Z_\circ)(\cos\beta x + j\sin\beta x)}{(Z_L + Z_\circ)(\cos\beta x - j\sin\beta x) - (Z_L - Z_\circ)(\cos\beta x + j\sin\beta x)} \right]$$

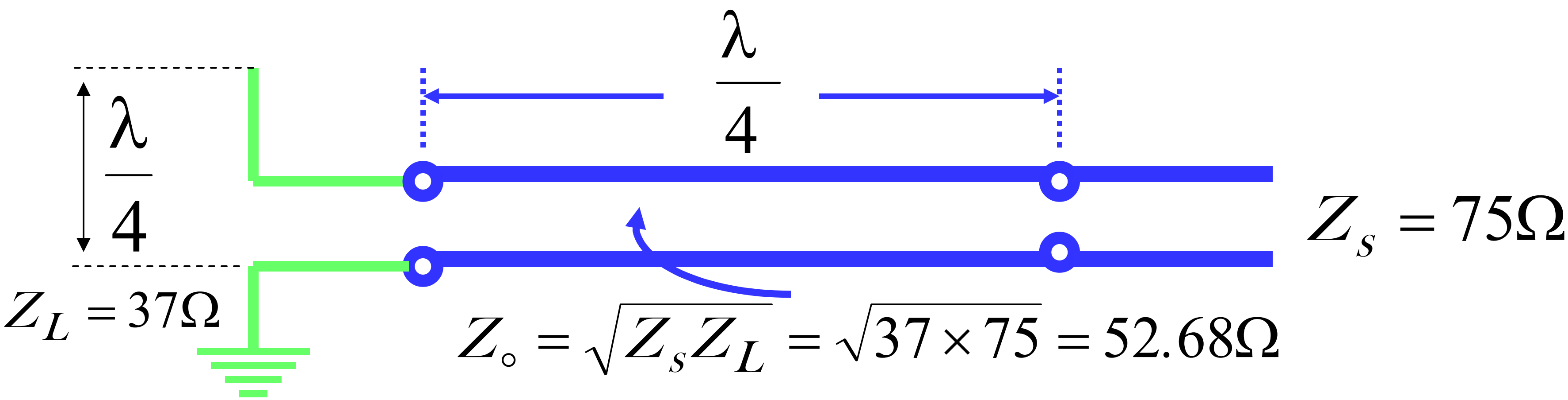
$$= Z_\circ \left[\frac{\begin{array}{l} Z_L\cos\beta x - jZ_L in\beta x + Z_\circ\cos\beta x - jZ_\circ\sin\beta x \\ + Z_L\cos\beta x + jZ_L\sin\beta x - Z_\circ\cos\beta x - jZ_\circ\sin\beta x \end{array}}{\begin{array}{l} Z_L\cos\beta x - jZ_L in\beta x + Z_\circ\cos\beta x - jZ_\circ\sin\beta x \\ - Z_L\cos\beta x - jZ_L\sin\beta x + Z_\circ\cos\beta x + jZ_\circ\sin\beta x \end{array}} \right]$$

$$= Z_\circ \left[\frac{2Z_L\cos\beta x - j2Z_\circ\sin\beta x}{2Z_L\cos\beta x - j2Z_L\sin\beta x} \right] \Rightarrow Z(x) = Z_\circ \frac{Z_L - jZ_\circ\tan\beta x}{Z_\circ - jZ_L\tan\beta x}$$

Impedance "visible" at end of line $Z(-\ell)$

$$Z(-\ell) = Z_\circ \frac{Z_L + jZ_\circ\tan\beta\ell}{Z_\circ + jZ_L\tan\beta\ell}$$

Example- Suppose we want to match a 75Ω transmission line to a 37 Ω Marconi Antenna.

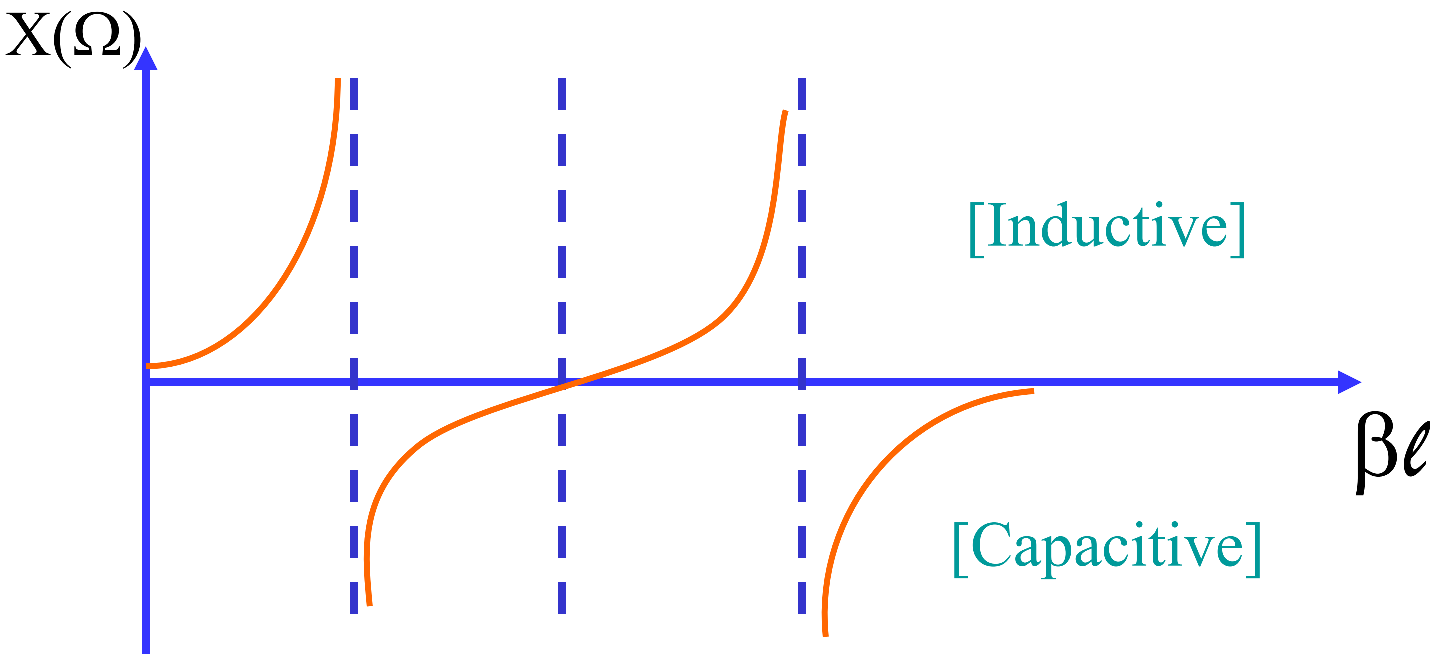


$$Z_\circ = \sqrt{Z_s Z_L} = \sqrt{37 \times 75} = 52.68\Omega$$

The matching cane therefore be achieved using a Quarter-Wavelength section of 50 Ω Transmission Line.

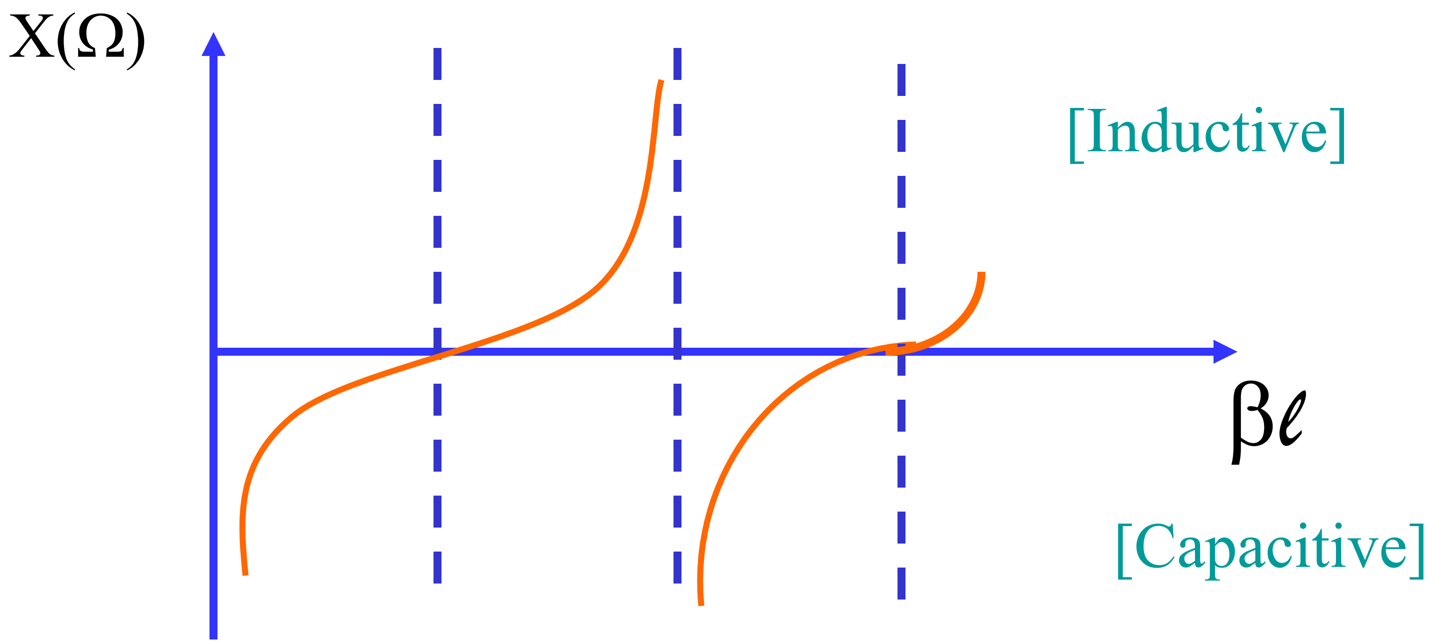
Now consider a short circuit termination: (Z_L=0)

$$\therefore Z(-\ell) = Z_\circ \frac{jZ_\circ \tan\beta\ell}{Z_\circ} = j \underbrace{Z_\circ \tan\beta\ell}_{Reactance\ X(\Omega)}$$



Now consider an open circuit termination: ($Z_L=\infty$)

$$\therefore Z(-\ell) = Z_\circ \frac{Z_L}{jZ_L \tan\beta\ell} = j(-Z_\circ \cot\beta\ell)$$



Problem

Suppose we have a load with an impedance of (40+j80)Ω which we need to match to a 100 Ω transmission line. (c 3x10^{8}m/s and f=130 MHz).

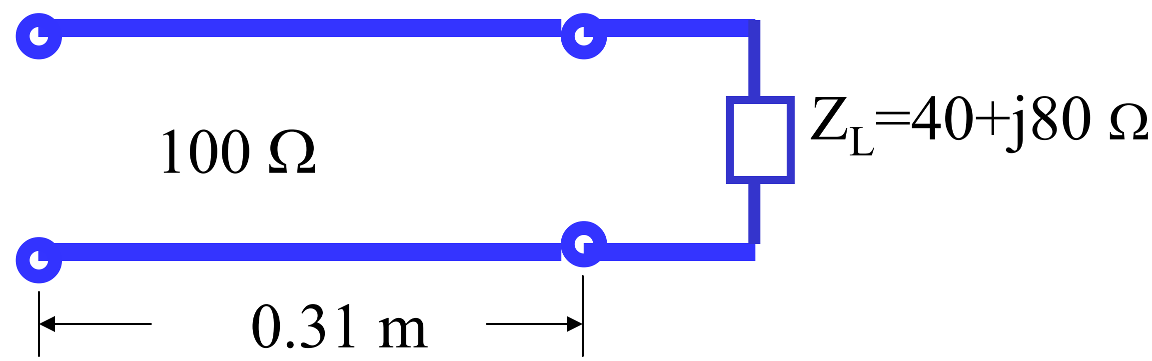
$$Z_{L(n)} = \frac{Z_L}{Z_\circ} = \frac{40 + j80\Omega}{100\Omega} = 0.4 + j0.8$$

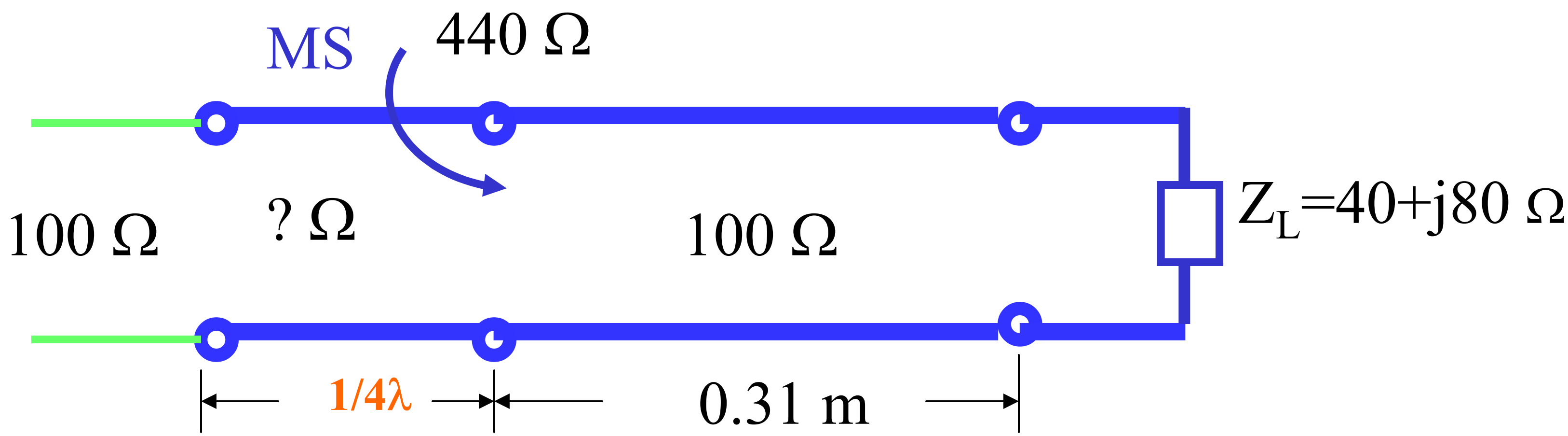
From the Smith Chart, the distance (in wavelength)

from this point toward the generator is $\sim 0.135\lambda$.

The distance is $d = 0.135 \times \dfrac{3\times10^8\, m/s}{130\times10^6\, Hz} = 0.31m$

Looking in here, we see a resistive load of 440 Ω . We can match this to 100 Ω using a 1/4λ matching section.





$$Z_{\circ} = \sqrt{Z_s Z_L}$$

$$Z_{\circ} = \sqrt{100 \times 440} = 209.8\Omega$$

Problem

Find the length, position and characteristic impedance of the quarter-wavelength transformer required to match an antenna with an impedance of (30-j40)Ω to a 75 Ω transmission line. The operating frequency of this system is 100MHz and the insulator in the transmission line has a dielectric constant of 10.

The speed of Wave Propagation $\nu = \frac{c}{\sqrt{\varepsilon_r}} = \frac{3\times10^8\, m/s}{\sqrt{10}} = 9.49\times10^7\, m/s$

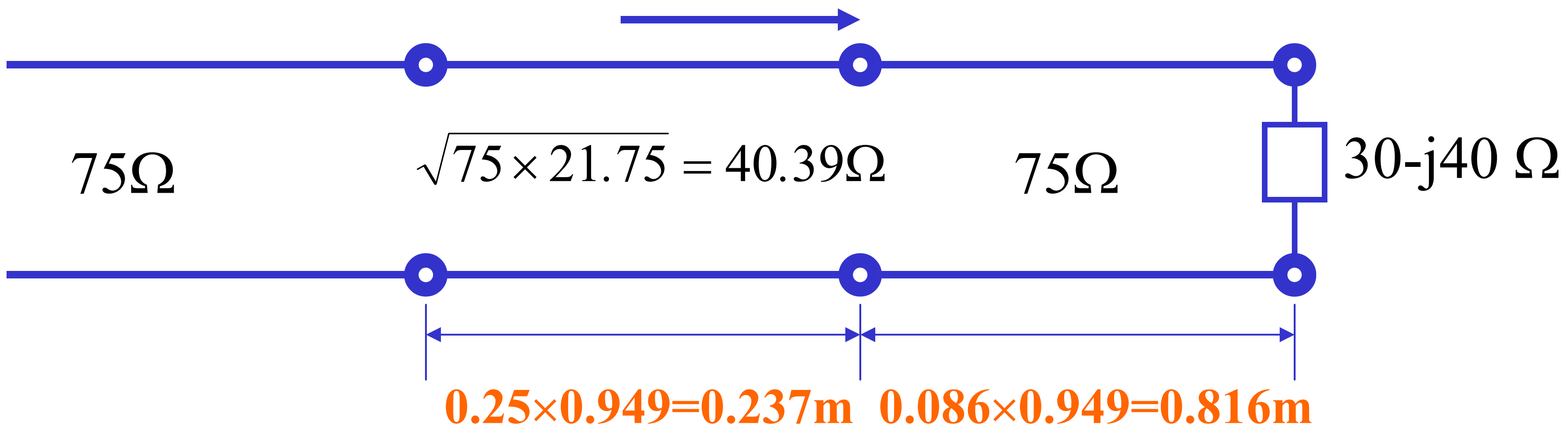
Wavelength $\lambda = \frac{\nu}{f} = \frac{9.49\times10^7}{100\times10^6} = 9.49\, m$

Normalized Impedance $Z_{L(n)} = \frac{Z_L}{Z_\circ} = \frac{30 - j40}{75} = 0.4 - j0.533$

Problem

From the Smith Chart, we find that we need to move 0.86λ away from the load (i.e., towards the generator) in order to eliminate the reactance (or imaginary) impedance component. At this point, the resistive (real) element is 0.29. Thus:

$$0.29 \times 75 = 21.75\Omega$$



Stub Matching :

We showed how a quarter-wavelength matching section can be used to impedance-match a load to a line. However, this technique has a major disadvantage: it is unlikely that one can find a transmission line with exactly the right characteristic impedance to perform the matching.

An alternative method is known as "Stub-Matching"

Using the parameters of example 1

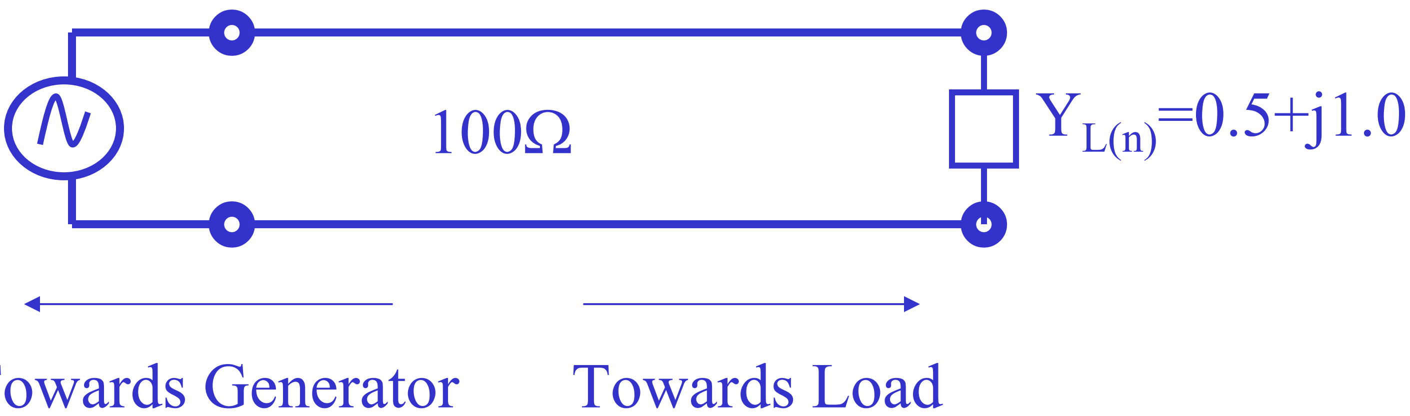
Suppose we have a load with an impedance of (40+j80)Ω which we need to match to a 100 Ω transmission line. (c 3×10^8m/s and f=130 MHz).

Stub Matching requires us to convert from impedance Z to admittance Y.

Stub Matching :

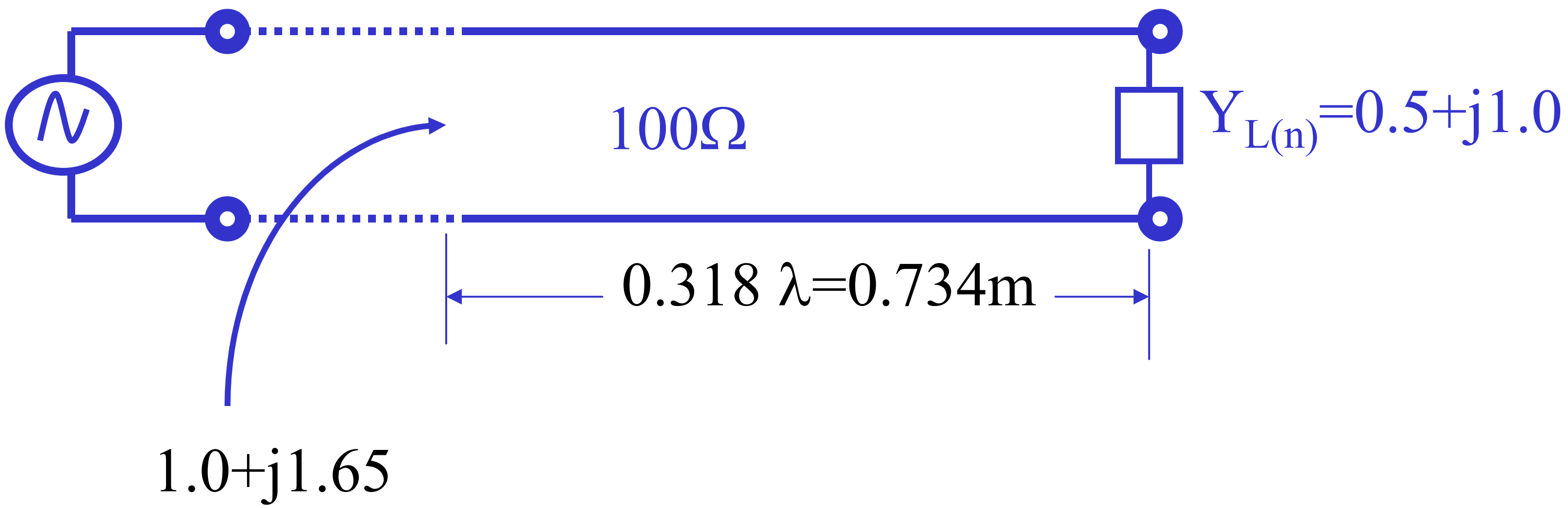
$$Y_{L(n)} = \frac{1}{Z_{L(n)}} = \frac{Y_L}{Y_\circ} = \frac{Z_\circ}{Z_L}$$

$$= \frac{100}{400 + j80} \times \frac{400 - j80}{400 - j80} = \frac{4000 - j8000}{\left(40^2 + 80^2\right)} = 0.5 - j1.0$$



Stub Matching :

$$\lambda = \frac{3\times10^8}{130\times10^6} = 2.3m \Rightarrow 0.318\lambda = 0.734m$$

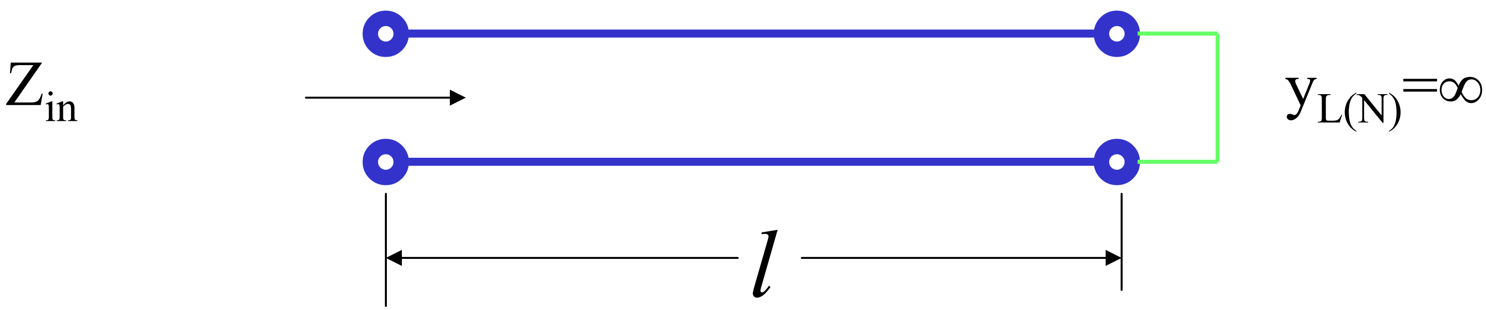


Looking into the line at the point X, we see a normalized admittance (1+j1.65).

If we introduce a reactive admittance of –j1.65 in parallel with the line at this point, the overall admittance will be 1.+j0 and the matching will be achieved.

Stub Matching :

A pure reactance can be created by means of a "Stub",i.e.,a length of transmission line with a short-circuit termination.

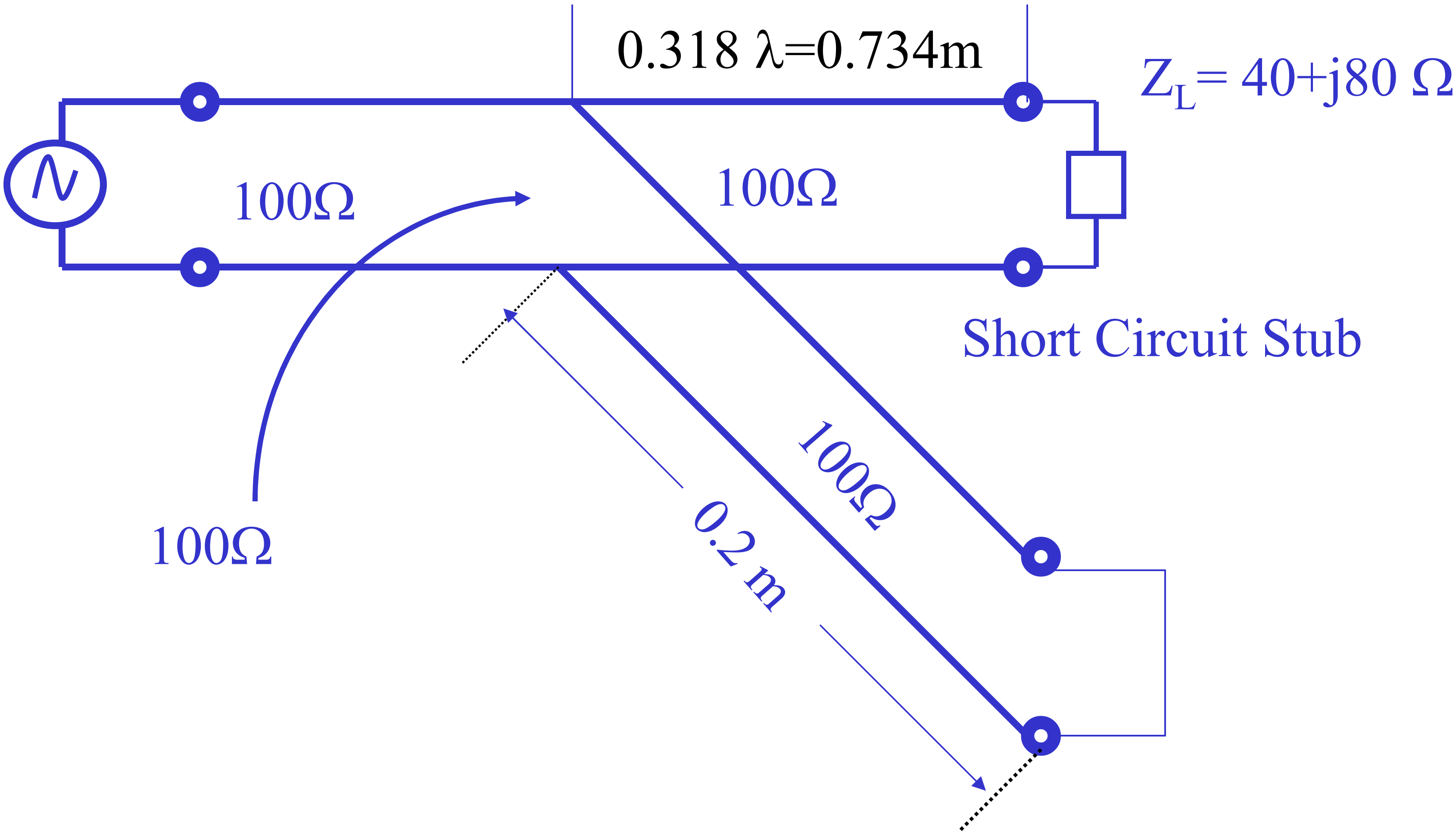


We showed before that the input impedance of a short-circuited line is given by

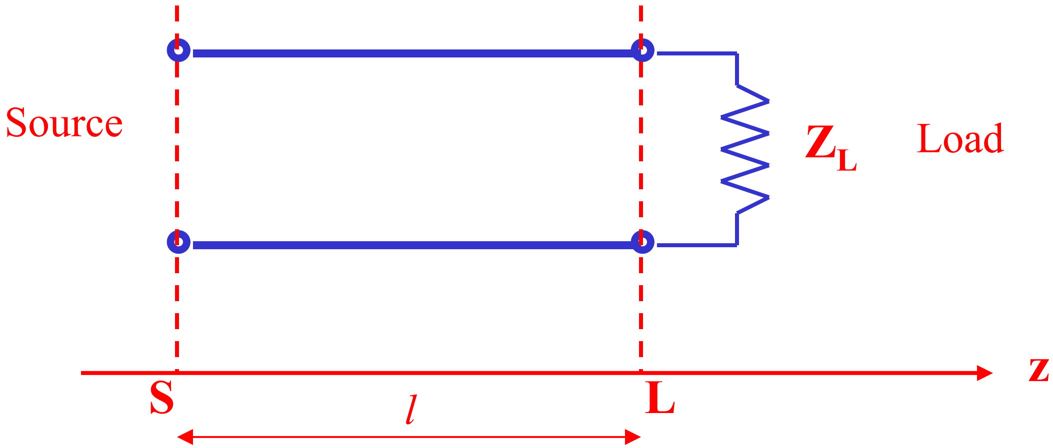
$$Z_{in} = jZ_{\circ}\tan\beta\ell$$

$$\therefore \textit{Input Admittance is } y_{in} = \frac{1}{jZ_{\circ}\tan\beta\ell} = -\frac{j}{Z_{\circ}}\cot\beta\ell$$

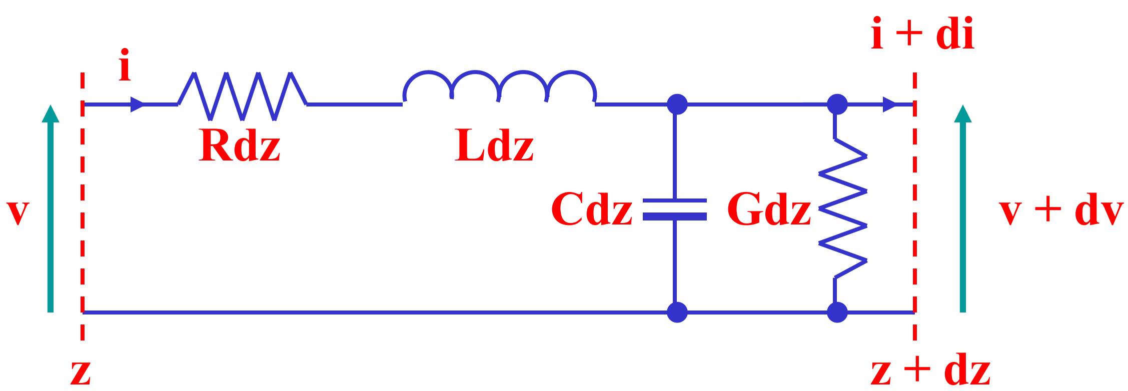
Stub Matching :



Transmission Line



We assume the line is characterized by distributed resistance R, inductance L, capacitance C, and conductance G per unit length.



Transmission Lines

Applying Kirchhoff's voltage law to the small length dz:

$$-dv = Rdzi + Ldz\frac{\partial i}{\partial t}$$

$$\frac{\partial v}{\partial z} = -Ri - L\frac{\partial i}{\partial t}$$

Applying Kirchhoff's current law to the small length dz:

$$-di = Gdz(v + dv) + Cdz\frac{\partial (v + dv)}{\partial t}$$

$$\frac{\partial i}{\partial z} = -Gv - C\frac{\partial v}{\partial t}$$

These basic equations are difficult to solve for non-sinusoidal waves in the general case. We only consider

1) transients on lossless lines

2) sinusoidal waves on lossy lines.

Transmission Lines (Lossless Lines)

In a lossless line we put R-0 and G=0.

$$\frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t}$$

$$\frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t}$$

If we eliminate *i* we obtain the one-dimensional wave equation

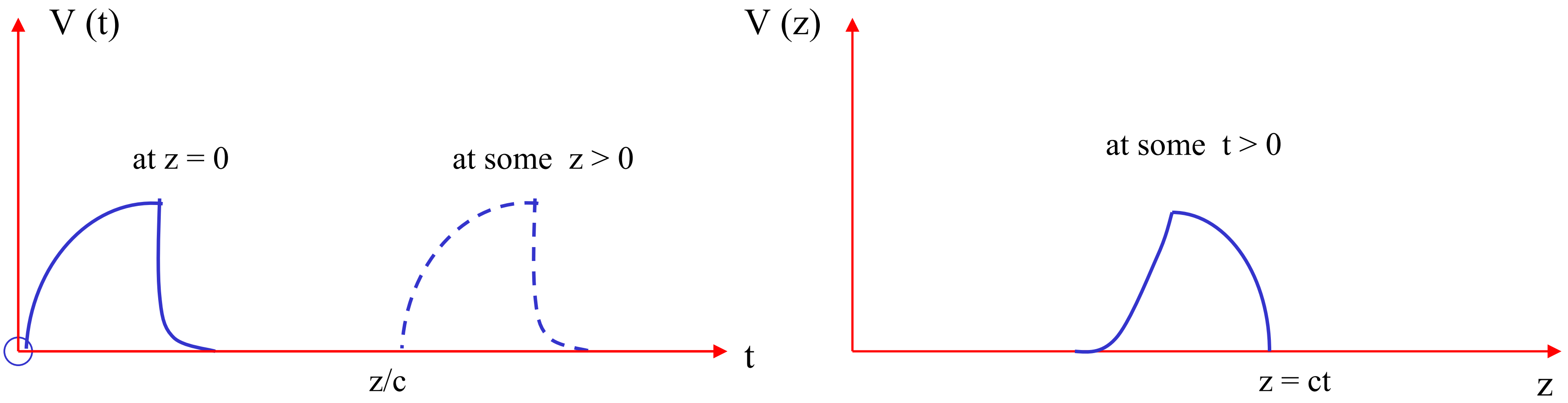
$$\frac{\partial^2 v}{\partial z^2} = LC\frac{\partial^2 v}{\partial t^2}$$

The general solution contains forward and reverse traveling waves of arbitrary shape, and has the form

$$v(z,t) = V_f(z-ct) + V_r(z+ct)$$

Exercise: Verity this solution satisfies the wave equation.

Transmission Lines (Lossless Lines)



Relation of voltage to current

$$\frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t}$$

Substitute v back into the equation and integrate with respect to time,

$$\sqrt{\frac{L}{C}}\, i(z,t) = V_f(z-ct) - V_r(z+ct) + f(z) \qquad f'(z)=0 \Rightarrow f(z)=const$$

$$Z_0 i(z,t) = V_f(z-ct) - V_r(z+ct), \quad Z_0 = \sqrt{\frac{L}{C}}$$

The current can be written as the sum of forward and revese components as

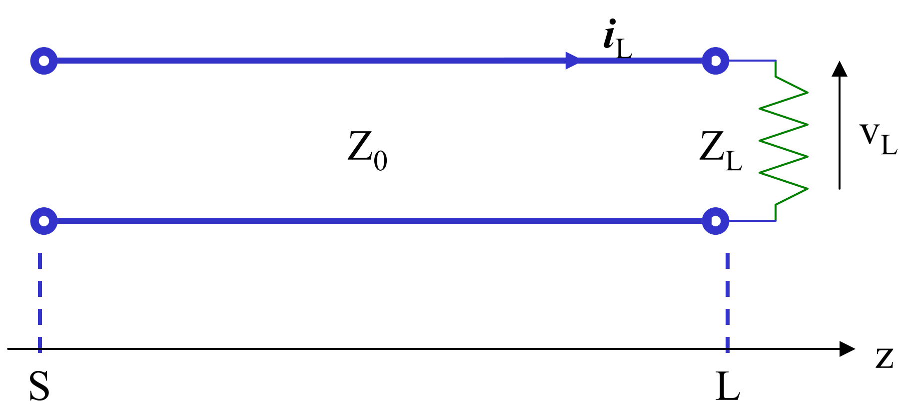
$$i(z,t) = I_f(z-ct) + I_r(z+ct)$$

where

$$I_f = \frac{1}{Z_0}V_f \qquad I_r = -\frac{1}{Z_0}V_r$$

Reflection

We consider a simple transmission line



At the load z =L we have

$$v = V_f + V_r = v_L$$

and

$$Z_0 i = V_f - V_r = Z_0 i_L$$

$$\Rightarrow \quad \frac{V_f - V_r}{V_f + V_r} = \frac{Z_0}{Z_L}$$

Re-arrange to obtain at the load:

$$\frac{V_r}{V_f} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Voltage reflection factor

We define the *voltage reflection factor* of the load as

$$\Gamma_v(L) = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Special cases

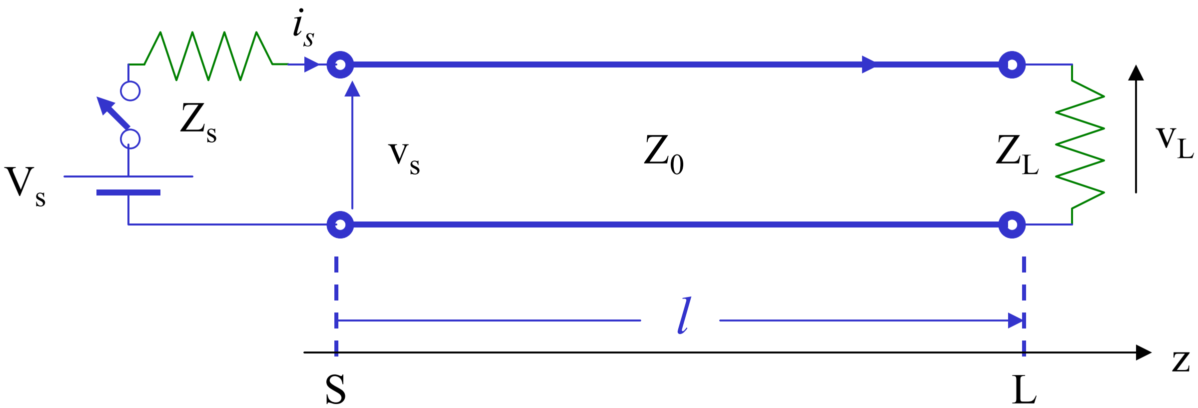
Condition	Z_L	$\Gamma_v(L)$
Matched	Z_0	0
o/c	∞	1
s/c	0	-1

The input resistance of the line when there is no reflected wave is

$$R_{in} = \frac{v_{in}}{i_{in}} = \frac{V_f}{I_f} = \frac{Z_0 I_f}{I_f} = Z_0$$

The input resistance of an initially uncharged line is initially equal to the characteristic impedance.

Example



At z = S

$$v_S = V_f(S,t) + V_r(S,t)$$

and

$$Z_0 i_S = V_f(S,t) - V_r(S,t)$$

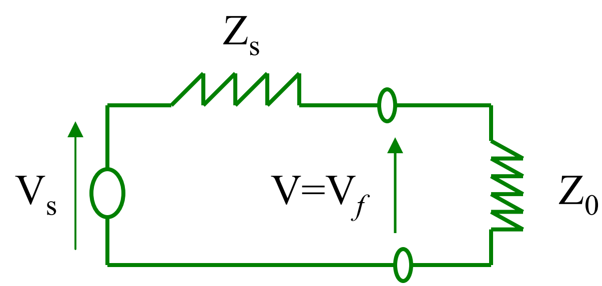
v_S and i_S must satisfy the boundary conditions provided by the source viz.

$$v_S = V_S - Z_S i_S$$

If we eliminate v_s and iS from the equations above, we obtain

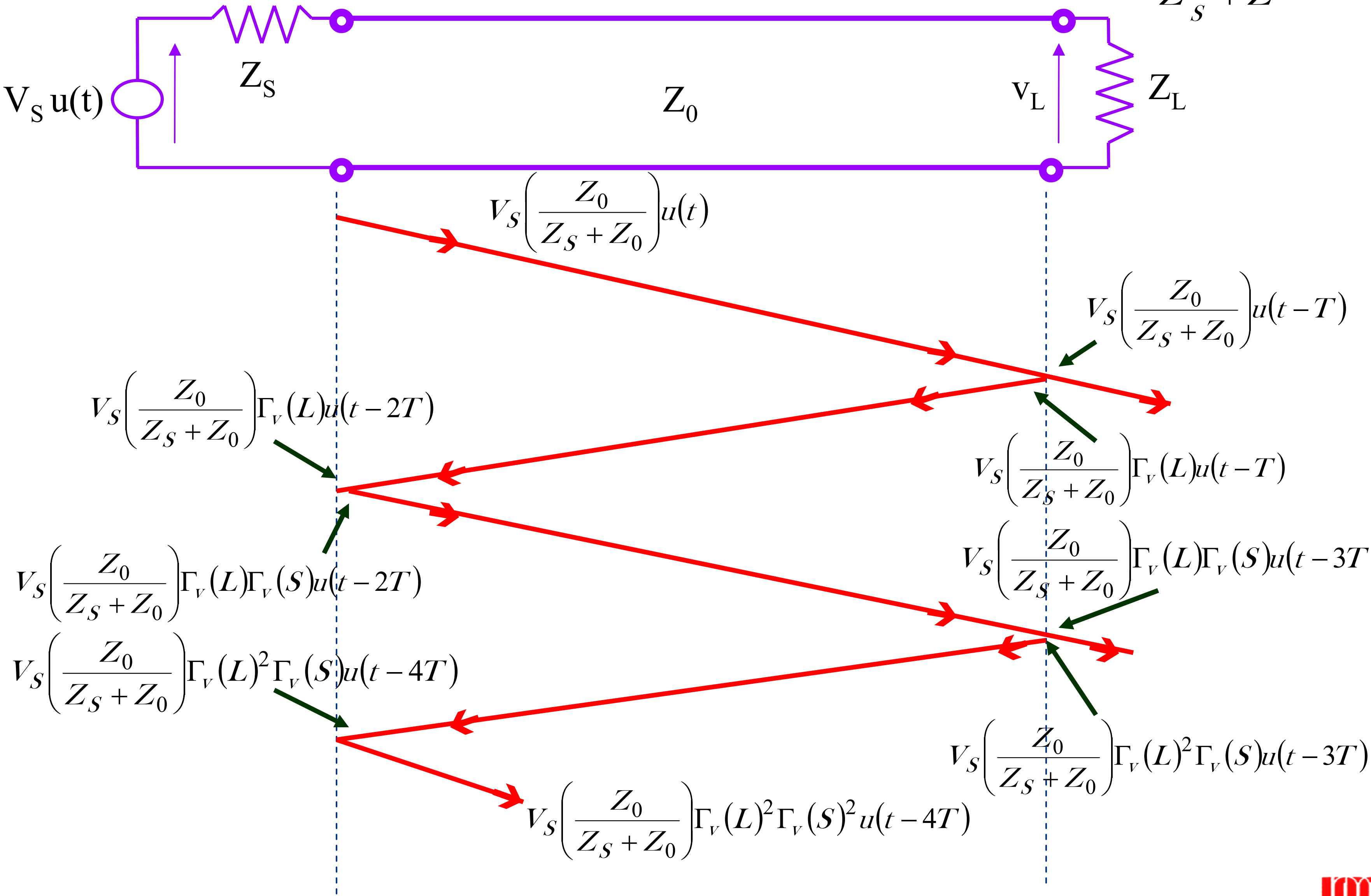
$$V_f(S,t) = V_S\left(\frac{Z_0}{Z_S + Z_0}\right) + V_r(S,t)\left(\frac{Z_S - Z_0}{Z_S + Z_0}\right)$$

$V_r(S, t) \equiv 0 \quad 0 \leq t < 2T$



Example

From the load end $V_r(L,t) = V_f(L,t)\Gamma_v(L)$, $\Gamma_v(L) = \dfrac{Z_L - Z}{Z_s + Z}$



Example

Lattice Diagram

$$v_L(t) = V_S\left(\frac{Z_0}{Z_S+Z_0}\right)\left[1+\Gamma_v(L)\right]\left[u(t-T)+\Gamma_v(L)\Gamma_v(S)u(t-3T)+\Gamma_v(L)^2\Gamma_v(S)^2u(t-5T)+\ldots\right]$$

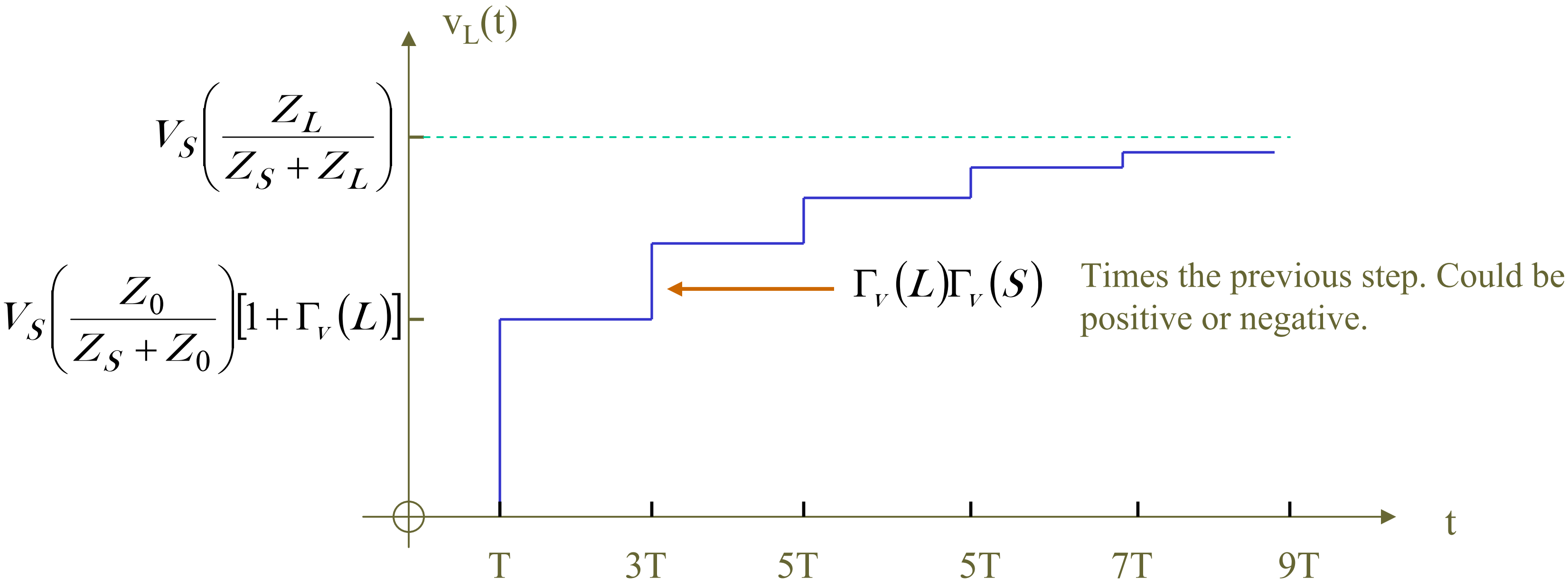
This has the form of a *geometrical progression* with common ration $\Gamma_v(L)\Gamma_v(S)$. For large values of t the final value of the load voltage may be shown by summing the geometrical progression above to be

$$v_L(t) \rightarrow V_S\left(\frac{Z_0}{Z_S+Z_0}\right)\left[1+\Gamma_v(L)\right]\frac{1}{1-\Gamma_v(L)\Gamma_v(S)}$$

If we substitute for $\Gamma_v(L)$ and $\Gamma_v(S)$ in terms of Z_S, Z_L and Z_0 and re-arrange, we get

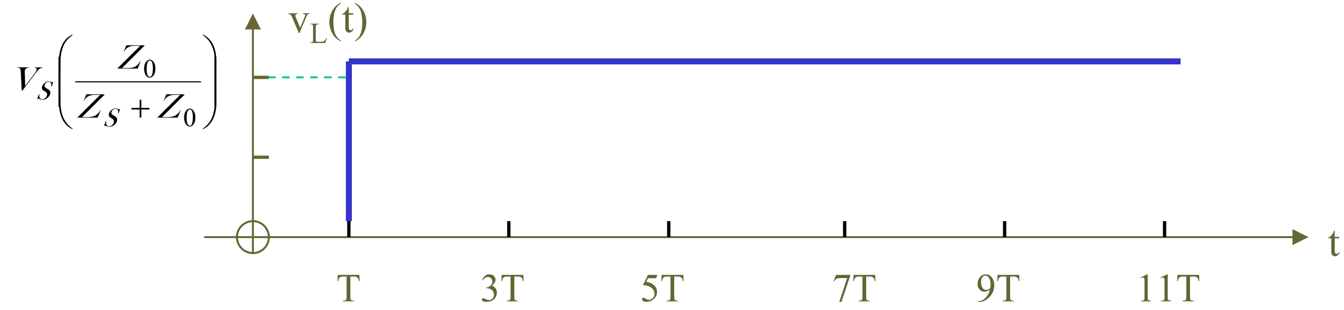
$$v_L(t) \rightarrow V_S\left(\frac{Z_L}{Z_S+Z_L}\right) \quad as \quad t \rightarrow \infty$$

Example

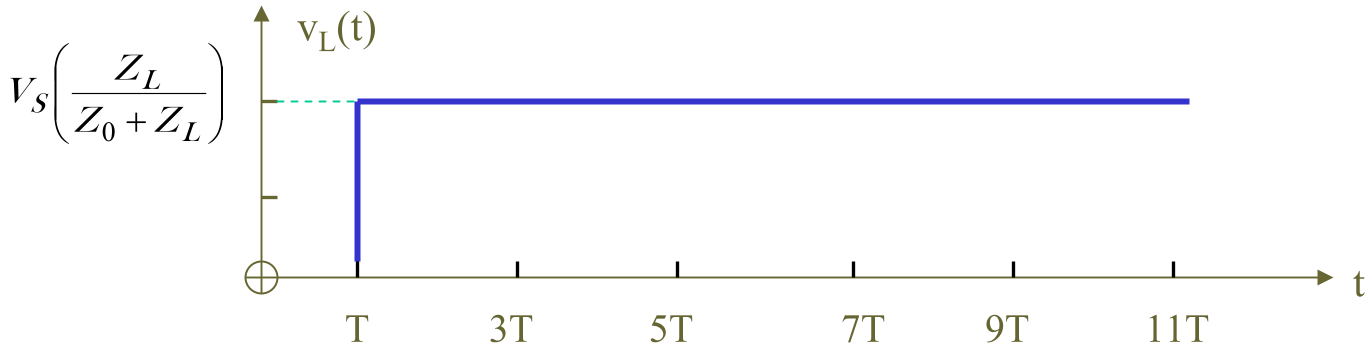


1. *No activity at the load until the time T*
2. *The initial step at that time is the product of two factors, initially launched forward wave on the line and the sum of the unity and the reflection factor at the load known as the transmission factor at the load junction.*
3. *Each of the subsequent steps is a common factor times the amplitude of the preceding step.*
4. *The steps become progressively smaller so that the eventual load voltage converges towards a value which is recognizable as the value the load voltage would have if one simply regarded the source impedance and load impedance as forming a voltage divider delivering to the load a fraction of the source voltage.*

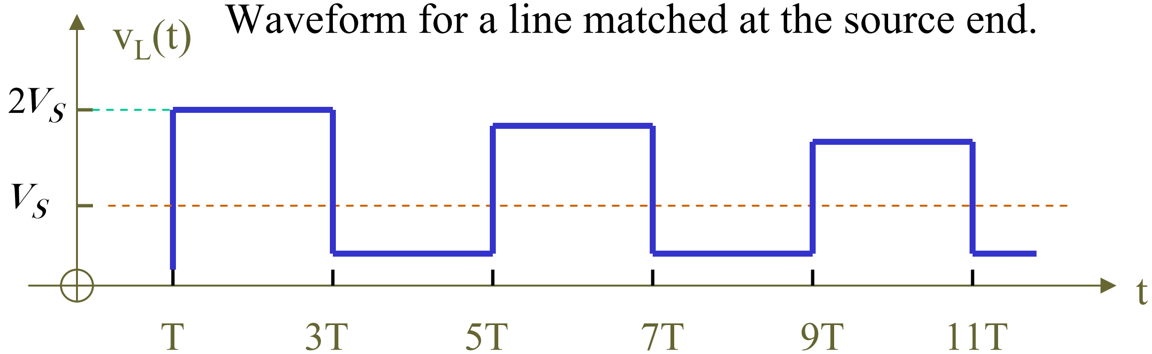
Special cases



Waveform for a line matched at the load end.

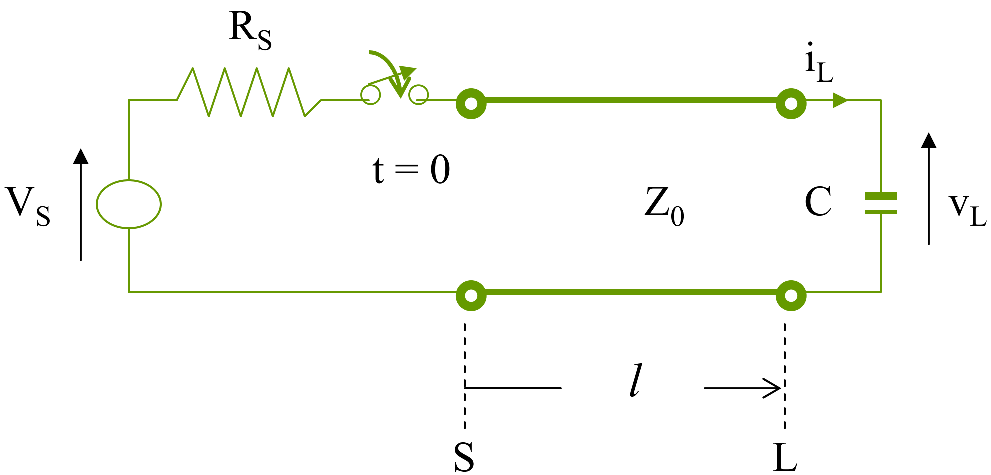


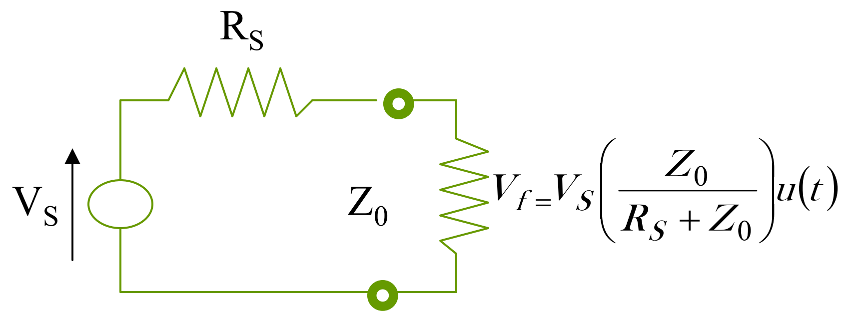
Waveform for a line matched at the source end.



Waveform for an open circuit line.

Non-resistive terminations



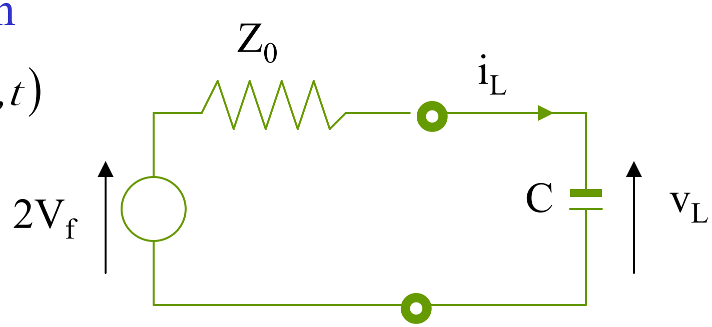


$$v_L = V_f(L,t) + V_r(L,t)$$

$$Z_0 i_L = V_f(L,t) - V_r(L,t)$$

Adding these, we obtain

$$v_L + Z_0 i_L = 2V_f(L,t)$$



for $0 \le t < 2T$

$$V_f = \frac{Z_0 V_S}{Z_0 + R_S} u(t - T)$$

With $\tau = Z_0 C$,

$$v_L = V_S u(t-T)\left(1 - e^{-(t-T)/\tau}\right) \quad \forall t$$

The reverse wave produced at the load end can then be found from

$$V_r(L,t) = v_L(t) - V_f(L,t)$$ and is

$$V_r(L,t) = V_S u(t-T)\left(\frac{1}{2} - e^{-(t-T)/\tau}\right) \forall t$$

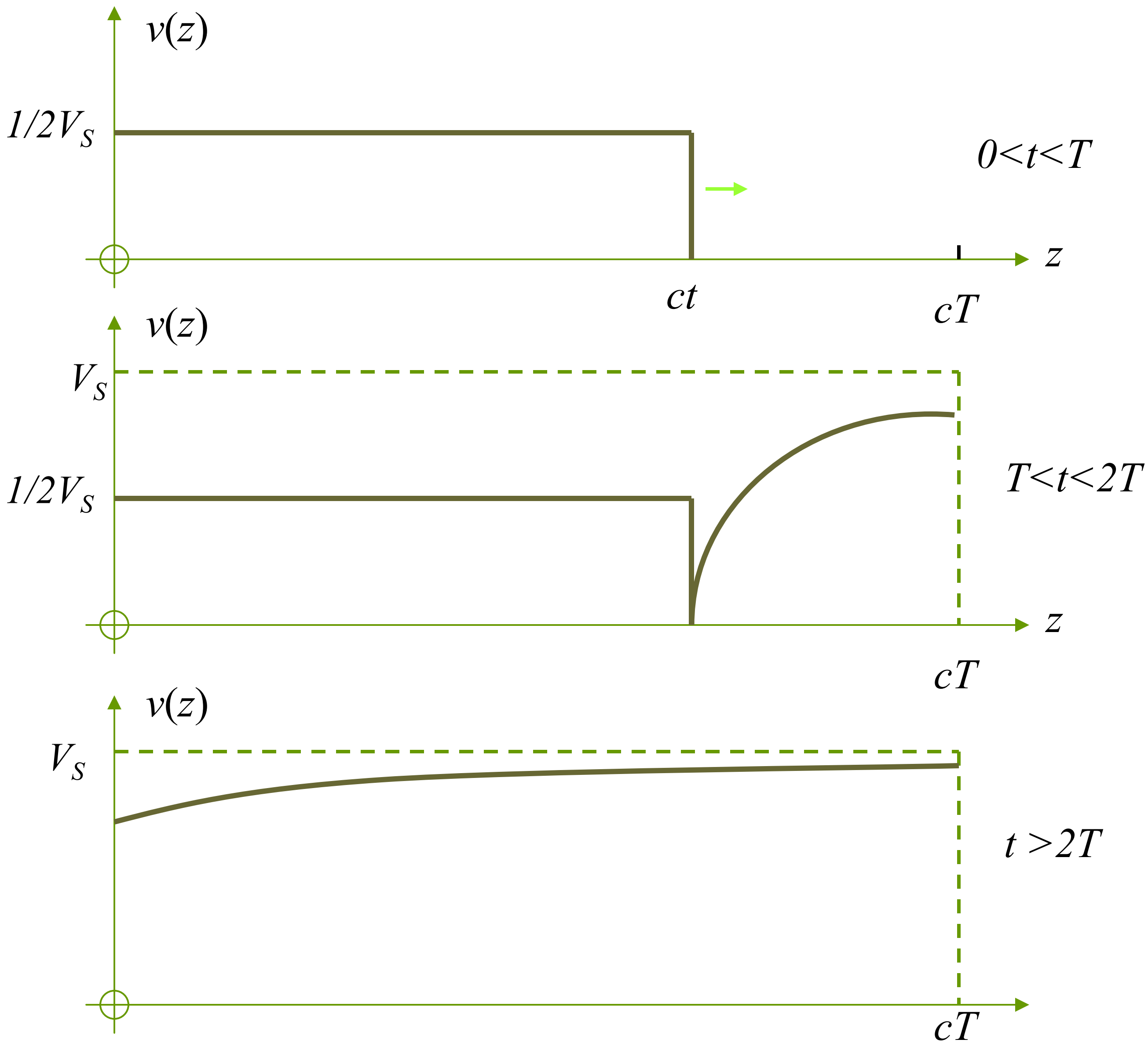
To find V_r at some point z<L we add a further delay time T-t' where $t' = z/c$ to obtain

$$V_r(z,t) = V_S u(t+t'-2T)\left(\frac{1}{2} - e^{-(t+t'-2T)/\tau}\right) \forall t$$

The total voltage on the line at any point and time is then obtained by adding to this backward wave the forward wave

$$V_f(z,t) = \frac{1}{2} V_S u(t-t')$$ to obtain

$$v(z,t) = \frac{1}{2} V_S \left(u(t-t') + u(t+t'-2T)\left(1 - 2e^{-(t+t'-2T)/\tau}\right)\right)$$



$v(z)$
$1/2V_S$
$0<t<T$
z
ct
cT
$v(z)$
V_S
$1/2V_S$
$T<t<2T$
z
cT
$v(z)$
V_S
$t>2T$
cT

Analysis in Frequency Domain

$$v(z,t) = \Re e\left\{V(z)e^{j\omega t}\right\}$$

V(z) is a *complex phasor* representing *peak* value.

$$\frac{dV}{dz} = -(R + j\omega L)I = -ZI \qquad Z = R + jX = R + j\omega L$$

$$\frac{dI}{dz} = -(G + j\omega C)V = -YV \qquad Y = G + jB = G + j\omega C$$

Solution:

Eliminating I

$$\frac{d^2V}{dz^2} = \gamma^2 V \qquad \gamma = \sqrt{ZY} = \sqrt{(R + j\omega L)(G + j\omega C)}$$

γ — *complex propagation constant*

$$V(z) = V_f e^{-\gamma z} + V_r e^{+\gamma z}$$

V_f represents the amplitude and the phase (at the origin) of a forward wave, while V_r represents the amplitude and phase (both at the origin) of a reverse wave).

$\gamma = \alpha + j\beta$ is the complex propagation constant.

- α: *attenuation constant*
- β: *phase constant*

The current *I z*) is

$$I(z) = -\frac{1}{Z}\left[-\gamma V_f e^{-\gamma z} + \gamma V_r e^{+\gamma z}\right]$$

Substituting for γ

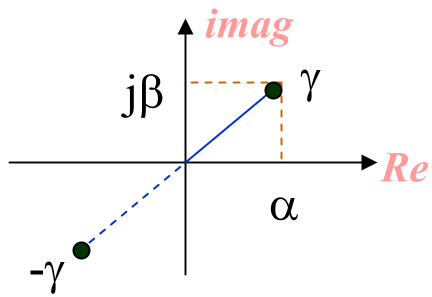
$$I(z) = \sqrt{\frac{Y}{Z}}\left[V_f e^{-\gamma z} - V_r e^{+\gamma z}\right]$$

$$= \frac{1}{Z_0}\left[V_f e^{-\gamma z} - V_r e^{+\gamma z}\right]$$

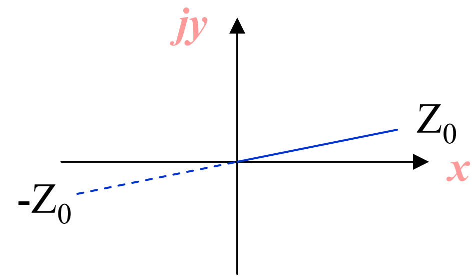
where we introduced

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

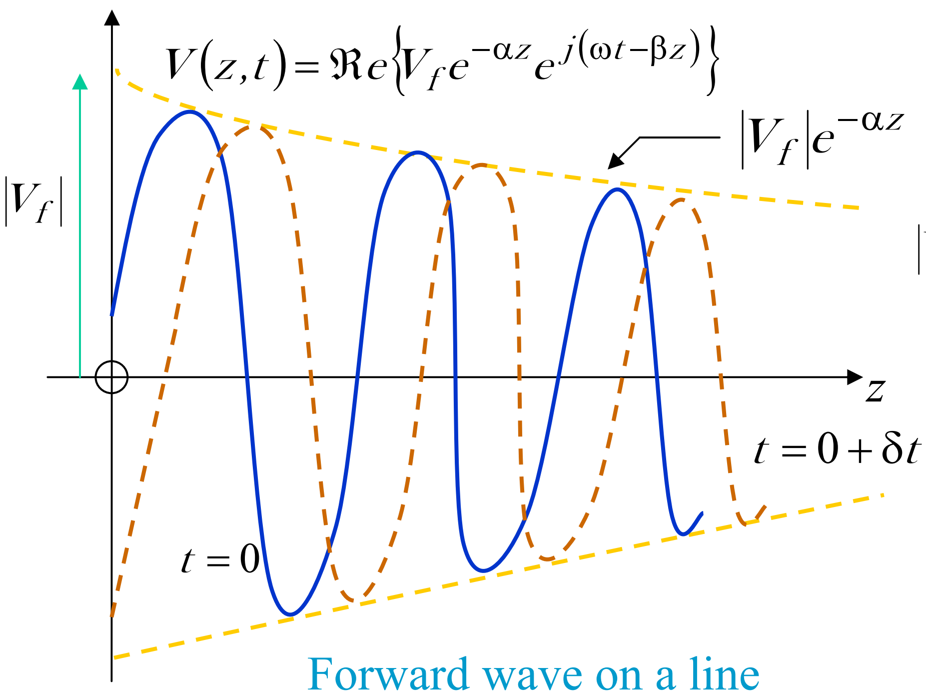
characteristic impedance of line



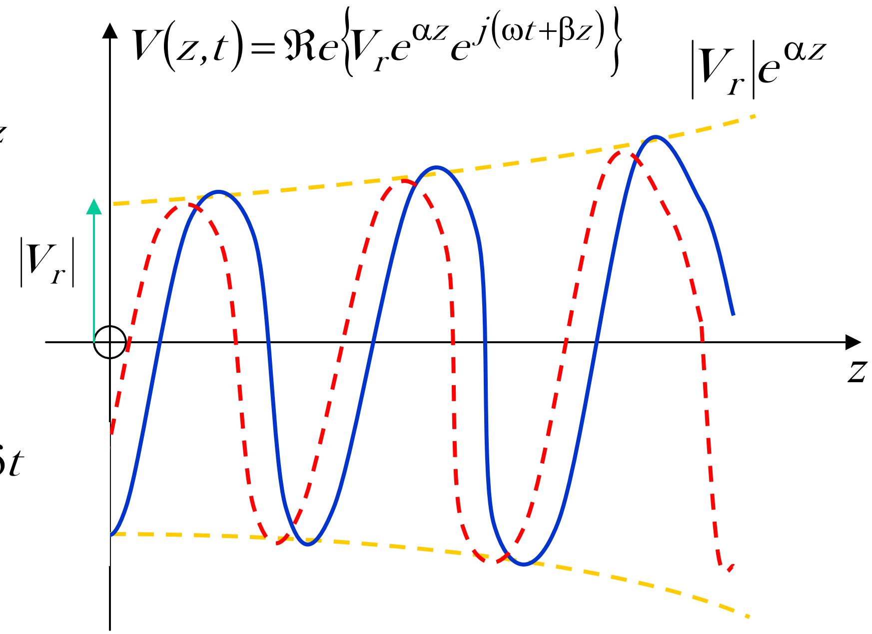
Argand Diagram for γ



Argand Diagram for Z_0



Forward wave on a line



Reverse wave on a line

We define the *complex voltage reflection factor* $\Gamma_v(z)$ at any point on the line as

$$\Gamma_v(z) = \frac{\textit{complex amplitude of the reverse voltage wave at z}}{\textit{complex amplitude of the forward voltage wave at z}}$$

$$\Gamma_V(z) = \frac{V_r e^{\gamma z}}{V_f e^{-\gamma z}} = \Gamma_V(0) e^{2\gamma z}$$

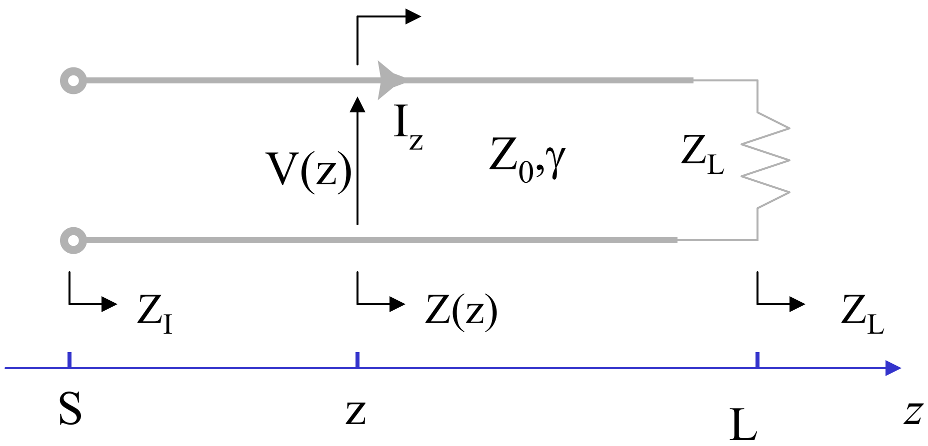
When z = L, i.e. at the load, we denote Γ_v by $\Gamma_v(L)$, the reflection factor of the load.
When z=S, i.e. at the source, we denote Γ_v by $\Gamma_v(S)$, *the reflection factor looking into the line* at the source end.

$$\frac{\Gamma_V(S)}{\Gamma_V(L)} = \frac{e^{2\gamma S}}{e^{2\gamma L}} = e^{-2\gamma(L-S)} = e^{-2\gamma \ell}$$

Impedance

We define *impedance at any point* by

$$Z(z) = \frac{V(z)}{I(z)} = \frac{V_f e^{-\gamma z} + V_r e^{+\gamma z}}{I_f e^{-\gamma z} + I_r e^{+\gamma z}}$$



$$\frac{Z(z)}{Z_0} = \frac{1+\Gamma_v(z)}{1-\Gamma_v(z)} \quad \Rightarrow \quad \Gamma_v(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0}$$

Combine steps to find Z_I

$$\frac{Z_I}{Z_0} = \frac{1 + \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) e^{-2\gamma\ell}}{1 - \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) e^{-2\gamma\ell}} \quad \Rightarrow \quad \frac{Z_I}{Z_0} = \frac{Z_L \cosh\gamma\ell + Z_0 \sinh\gamma\ell}{Z_0 \cosh\gamma\ell + Z_L \sinh\gamma\ell}$$

Matching

If a line is terminated in its characteristic impedance (which is complex for an arbitrary lossy line), i.e. if $Z_L = Z_0$, then

$$Z_I = Z_0 \text{ for any } l$$

Lossless TL

We assume R = 0 and G = 0:

$\alpha = 0$ i.e. no attenuation

$\beta = \omega\sqrt{LC}$ i.e. no dispersion

$$\nu_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = \text{constant}$$

$$\nu_g = \frac{\partial\omega}{\partial\beta} = \frac{1}{\sqrt{LC}} = \text{same constant}$$

Z_0 in now *real and independent of frequency:* $Z_0 = \sqrt{\frac{L}{C}}$

We find that the voltage reflection factor

$$\Gamma_V(z) = \Gamma_V(0)e^{2j\beta z}$$

changes in phase nut not in magnitude as we go along the line. It *advances* in phase along $+z$ direction *toward* the load. At the source

$$\Gamma_V(S) = \Gamma_V(L)e^{-2j\beta\ell}$$

distance *l* back from the load the voltage reflection factor is *retarded* in phase as we make the line longer and move *back* from the load.

$$\frac{Z_I}{Z_0} = \frac{Z_L \cos\beta\ell + jZ_0 \sin\beta\ell}{Z_0 \cos\beta\ell + jZ_L \sin\beta\ell}$$

Special cases

Case	Impedance
Short circuit load	$Z_I = jZ_0 \tan\beta\ell$
Shorted λ/4 line	$Z_I \to \infty$ i.e. o/c
Open circuit load	$Z_I = -jZ_0 \cot\beta\ell$
Open circuit λ/4 line	$Z_I = 0$ i.e. s/c

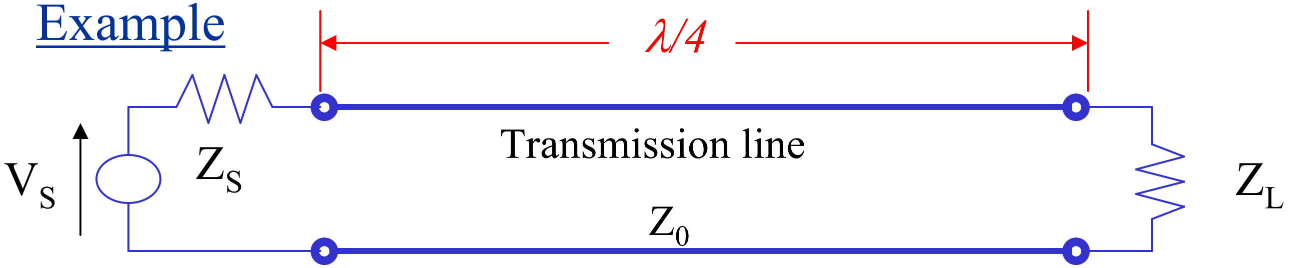
Quarter wave lines

When $l = \lambda/4$, $\beta l = \pi/2$. Then

$$Z_I = \frac{Z_0^2}{Z_L}$$

This important result means $\lambda/4$ lines can be used as *transformers*.

Example



$$Z_0 = \sqrt{Z_I Z_L} \longrightarrow Z_0 = \sqrt{(100\Omega)(50\Omega)} = 70.7\Omega$$

Normalized impedance

$$z = \frac{Z}{Z_0} \qquad \Gamma_V = \frac{z-1}{z+1} \qquad z = \frac{1+\Gamma_V}{1-\Gamma_V}$$

Admittance Formulation

For every impedance Z we have a corresponding admittance Y=1/Z. It is easy to show that

$$\frac{Y_I}{Y_0} = \frac{Y_L\, cos\,\beta\ell + jY_0\, sin\,\beta\ell}{Y_0\, cos\,\beta\ell + jY_L\, sin\,\beta\ell}$$

Special cases

Case	Impedance
Open circuit load	$Y_I = jY_0\, tan\,\beta\ell$
Open$\lambda/4$ line	$Y_I \to \infty$ i.e. s/c
Short circuit load	$Y_I = -jY_0\, cot\,\beta\ell$
Short circuit $\lambda/4$ line	$Y_I = 0$ i.e. o/c

Quarter wave lines

$$Y_I = \frac{Y_0^2}{Y_L}$$

Normalized admittance

$$y = \frac{Y}{Y_0} \qquad -\Gamma_v = \frac{y-1}{y+1} \qquad y = \frac{1-\Gamma_v}{1+\Gamma_v}$$

Current reflection factor

$$\Gamma_i(z) = \frac{I_r e^{\gamma z}}{I_f e^{-\gamma z}}$$

Substituting for I_f and I_r in terms of V_f and V_r, we find:

$$\Gamma_i = -\Gamma_v$$

$$\Gamma_i = \frac{y-1}{y+1}$$

$$y = \frac{1+\Gamma_i}{1-\Gamma_i}$$

Then

$$\Gamma_i(z) = \Gamma_i(0) e^{2j\beta z}$$

or

$$\Gamma_i(S) = \Gamma_i(L) e^{-2j\beta\ell}$$

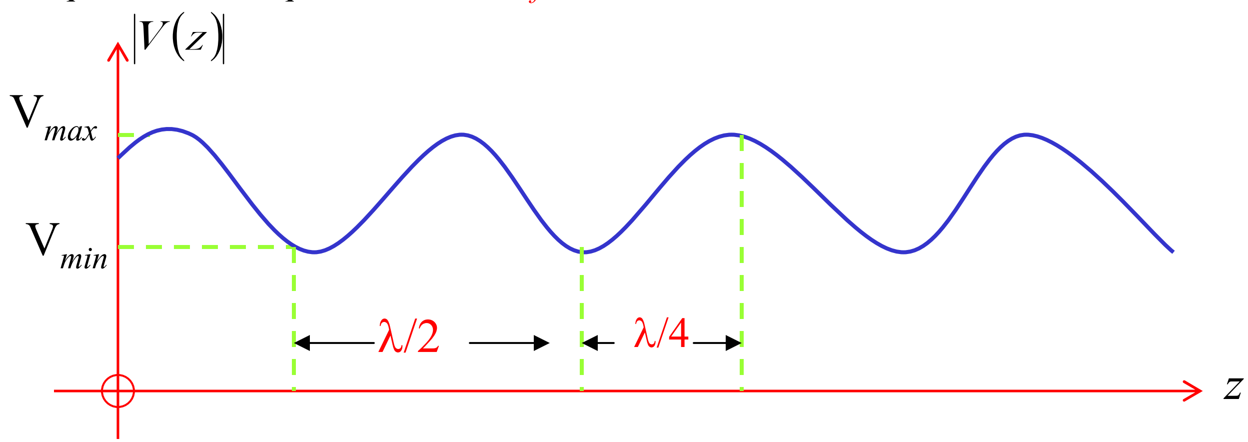
Voltage Standing Wave Ratio

We look at the way the total voltage V(z) varies along the lossless line. We have, with

$\gamma = j\beta$

$$V(z) = V_f e^{-j\beta z} + V_r e^{+j\beta z}$$

V_f and V_r are complex numbers



$$V_{max} = |V_f| + |V_r|$$

$$V_{min} = |V_f| - |V_r|$$

VSWR: $$S = \frac{V_{max}}{V_{min}} = \frac{|V_f| + |V_r|}{|V_f| - |V_r|}$$

$$|\Gamma_V| = \frac{S-1}{S+1}$$

$$S = \frac{1+|\Gamma_V|}{1-|\Gamma_V|}$$

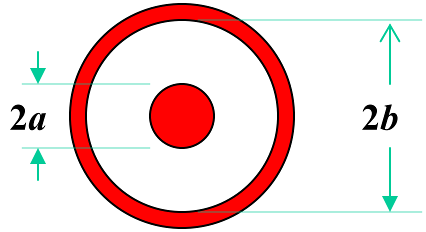
Line Parameters

Maximum and minimum values of impedance along the line can be related simply to S. When $V_f e^{-j\beta z}$ is in phase with $V_r e^{+j\beta z}$ we have a simultaneous voltage maximum and current minimum. Thus

$$Z_{max} = \frac{V_{max}}{I_{min}} = \frac{|V_f| + |V_r|}{\left(|V_f| - |V_r|\right)/Z_0} = SZ_0$$

and

$$Z_{min} = \frac{V_{min}}{I_{max}} = \frac{|V_f| - |V_r|}{\left(|V_f| + |V_r|\right)/Z_0} = \frac{Z_0}{S}$$



$$L = \frac{\mu_0}{2\pi} ln\left(\frac{b}{a}\right)$$

$$C = \frac{2\pi\varepsilon}{ln\left(\frac{b}{a}\right)}$$

$$\Rightarrow \quad Z_0 = \sqrt{\frac{L}{C}} = \frac{1}{2\pi}\sqrt{\frac{\mu_0}{\varepsilon}}\, ln\left(\frac{b}{a}\right)$$

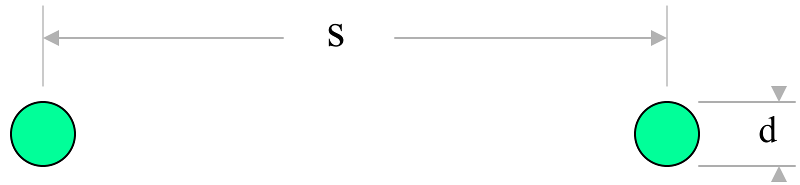
The complex propagation constant is

$$\alpha + j\beta = \sqrt{YZ} = j\omega\sqrt{LC}$$

There is no attenuation since we assumed there are no losses and the velocity $c = \omega / \beta$ is

$$c = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

Consider a twin line



$$L = \frac{\mu_0}{\pi}\, ar\cosh\left(\frac{s}{d}\right) \approx \frac{\mu_0}{\pi}\, ln\left(\frac{2s}{d}\right) \qquad s >> d$$

$$C = \frac{\pi\varepsilon}{ar\cosh\left(\frac{s}{d}\right)} \approx \frac{\pi\varepsilon}{ln\left(\frac{2s}{d}\right)}$$

$$Z_0 = \frac{1}{\pi}\sqrt{\frac{\mu_0}{\varepsilon}}\, ln\left(\frac{2s}{d}\right) \qquad s >> d$$

Common values of Z_0 are 300 Ω for communication lines, 600 Ω for telephone lines and slightly higher values are found for power lines.

Matching of T.L.

We recall from lumped circuit theory the *maximum power transfer theorem* for a.c. circuits which indicates that a sinusoidal steady state source of fixed internal voltage V_s and source impedance Z_S will deliver maximum power to a load impedance Z_L when Z_L is adjusted to be the *complex conjugate* of the source impedance Z_S, that is

$$Z_L = Z_S^*$$

Matching of the T.L. *at both ends* makes power transfer between the source and the load take place at minimum loss, and also makes the system behavior become independent of the line length.