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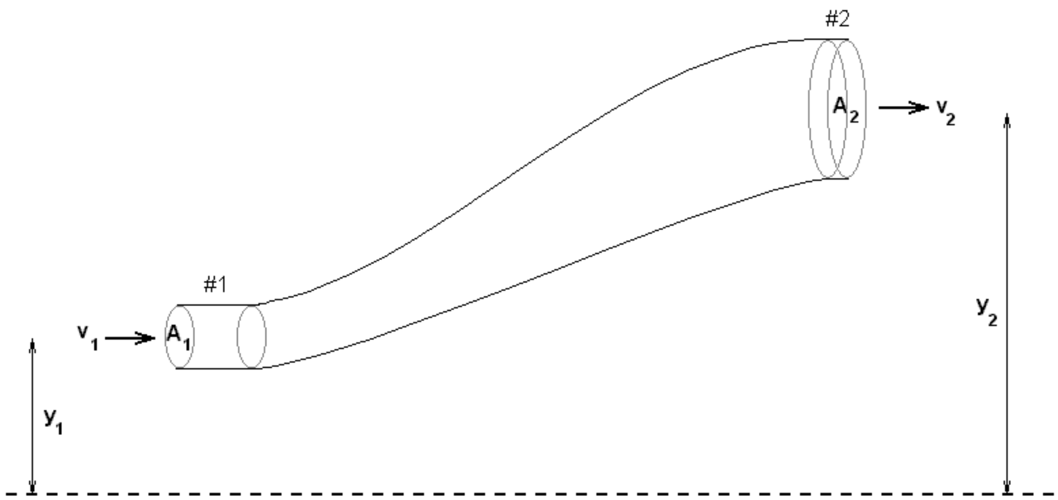
Physics: An Incremental Development, John H. Saxon, Jr.

Lesson 61 - Derivation of Bernoulli's Equation

We focus on the case of an incompressible fluid with density ρ (= a constant) moving with a constant, non-turbulent volume flow rate (= $A_1 \cdot v_1$ m^3/s = $A_2 \cdot v_2$ m^3/s = a constant). The pressure is also constant as long as the radius and elevation are constant. The radius of the pipe and the elevation of the pipe may change, however.

We will follow the pressure changes in a volume of fluid (volume element) flowing first at point #1, then at point #2. The same volume flow rate is observed at both points in the pipe by our assumption. We choose two points where the pipe and thus the velocity of the fluid are horizontal to simplify the math without changing the general conclusions. The following diagram summarizes the general case.



The total mechanical energy of the volume element at position #1, E_1, is

$$E_1 = \text{kinetic energy} + \text{potential energy} = \tfrac{1}{2} \cdot \Delta m_1 \cdot v_1^2 + \Delta m_1 \cdot g \cdot y_1$$

(where v_1 is velocity, y_1 is elevation, Δm_1 is mass)

Similarly for the volume element at position #2, its total mechanical energy, E_2, is

$$E_2 = \text{kinetic energy} + \text{potential energy} = \tfrac{1}{2} \cdot \Delta m_2 \cdot v_2^2 + \Delta m_2 \cdot g \cdot y_2$$

(where v_2 is velocity, y_2 is elevation, Δm_2 is mass)

Under our assumed conditions of constant volume flow for an incompressible fluid, the mass of the volume elements is conserved during the flow, so that we may substitute

$$\Delta m_1 = \Delta m_2 = \Delta m$$

Clearly, if the flow was steady ($v_1 = v_2 = v$), and if there was no change in the elevation ($y_1 = y_2 = y$), then there would be no change in the total mechanical energy of the fluid. If, however, either one or both of those parameters changes between position #1 and position #2, there will be changes in the total mechanical energy of the fluid, as follows:

$$E_2 - E_1 = ½\bullet\Delta m\bullet v_2^2 + \Delta m\bullet g\bullet y_2 - ½\bullet\Delta m\bullet v_1^2 - \Delta m\bullet g\bullet y_1$$

This difference in mechanical energy arises from the Net Work done on the system, ie, the Net Work done on all the fluid in the pipe between point #1 and point #2.

(Net Work = Work done on the fluid <u>minus</u> work done by the fluid = $W_{on} - W_{out}$)

At point #1, W_{on} = [force]•[distance] = [pressure • area]•[velocity • time interval]

$$= (P_1\bullet A_1)\bullet(v_1\bullet\Delta t)$$ (Δt is time to move an element length)

$$= P_1\bullet(\rho\bullet A_1\bullet v_1\bullet\Delta t) / \rho$$ (ρ/ρ inserted)

But $(\rho\bullet A_1\bullet v_1\bullet\Delta t)$ is the mass of our volume element at position #1. Note that $(v_1\bullet\Delta t)$ is simply the length of the volume element; A_1 is its cross-sectional area. Those two multiplied together give us its volume. Density times volume equals mass. Therefore,

$$W_{on} = P_1\bullet \Delta m / \rho$$

The same arguments apply at point #2. The work done by the system will be subtracted because this work decreases the total mechanical energy of the fluid.

$$W_{out} = P_2 \bullet \Delta m / \rho$$

Thus, the net work done on the fluid, which is the work done on the system minus the work done by the system, is

$$W_{on} - W_{out} = P_1 \bullet \Delta m / \rho - (P_2 \bullet \Delta m / \rho) = (P_1 - P_2) \bullet \Delta m / \rho$$

This Net Work is the source of the energy difference, $E_2 - E_1$, which we can rearrange as follows:

$$E_2 - E_1 = \Delta m \bullet \{½ \bullet (v_2^2 - v_1^2) + g \bullet (y_2 - y_1)\}$$

Now, the Net Work and the change in mechanical energy must equal each other (Work-Energy Theorem), therefore,

$$(P_1 - P_2) \bullet \Delta m / \rho = \Delta m \bullet \{\tfrac{1}{2} \bullet (v_2^2 - v_1^2) + g \bullet (y_2 - y_1)\}$$

The mass terms cancel, because the fluid is incompressible, and we can move the density to the other side of the equation, leaving,

$$P_1 - P_2 = \rho \bullet \{\tfrac{1}{2} \bullet (v_2^2 - v_1^2) + g \bullet (y_2 - y_1)\}$$

$$= (\tfrac{1}{2}\bullet\rho\bullet v_2^2) - (\tfrac{1}{2}\bullet\rho\bullet v_1^2) + (\rho\bullet g\bullet y_2) - (\rho\bullet g\bullet y_1)$$

Or, after moving all negative terms to the opposite side of the equation,

$$P_1 + (\tfrac{1}{2}\bullet\rho\bullet v_1^2) + (\rho\bullet g\bullet y_1) = P_2 + (\tfrac{1}{2}\bullet\rho\bullet v_2^2) + (\rho\bullet g\bullet y_2)$$

Which gives us **Bernoulli's Equation** in its normal presentation as

$$P_1 + \tfrac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \tfrac{1}{2}\rho v_2^2 + \rho g y_2 = \text{constant everywhere in the fluid}$$

Interpretation of Bernoulli's Equation

We should first note that, if the cross-sectional areas at points #1 and #2 are the same, then the two velocities are equal and the $\frac{1}{2}\rho v^2$-terms cancel each other. Further, if there is no difference in the heights at points #1 and #2, the two ρgy-terms cancel each other. If both are true, then no matter what happens to the pipe size and elevation between these two points, the pressure at these two points will still be equal; $P_1 = P_2$.

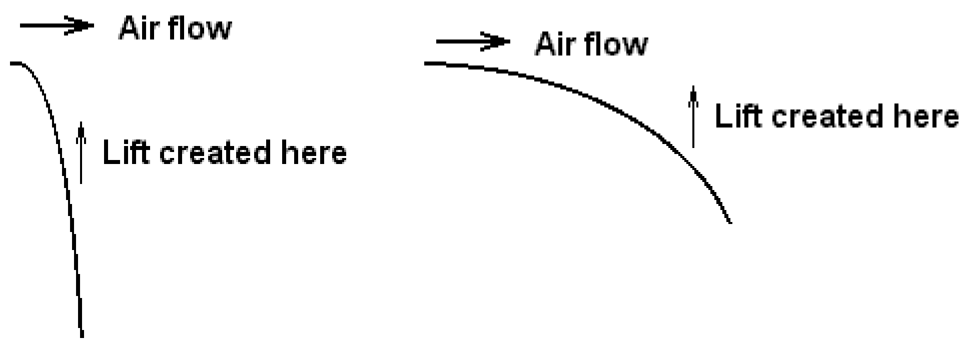
If we examine the case where only the elevation is constant, then we get

$$P_1 + \tfrac{1}{2}\rho v_1^2 = P_2 + \tfrac{1}{2}\rho v_2^2$$

This takes a little contemplation, but what this equation tells us is that as the velocity increases the pressure decreases. Think of it this way: the two sides of the equation must remain equal. If v_2 increases because of a decrease in A_2, then P_2 must decrease in order for their sum to maintain its equality with the left side of the equation.

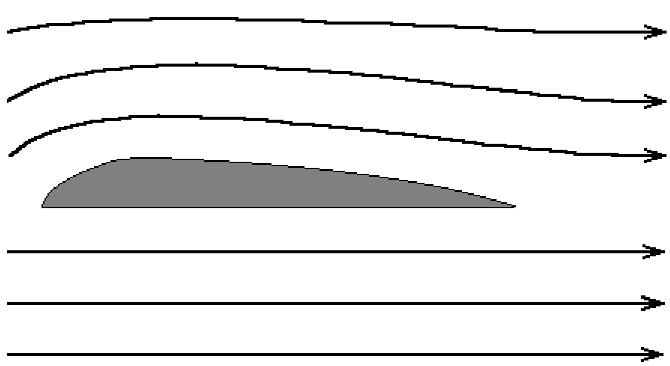
It is easy to demonstrate that this is true. The mathematics is elegant and simple, but sometimes you just have to check with the universe to make sure it agrees with you. So, try this little experiment for yourself.

Hold a piece of paper bent-over and hanging vertically from just below your lips. Blow horizontally just above the top of the bent leading edge of the paper. Do not blow under the paper; that would be easier but not in the spirit of Bernoulli's Equation. Like this…



Applications of Bernoulli's Equation

<u>Operation of an airplane wing:</u>



The shape of the wing forces the air moving above the wing to move faster than the air moving below the wing.

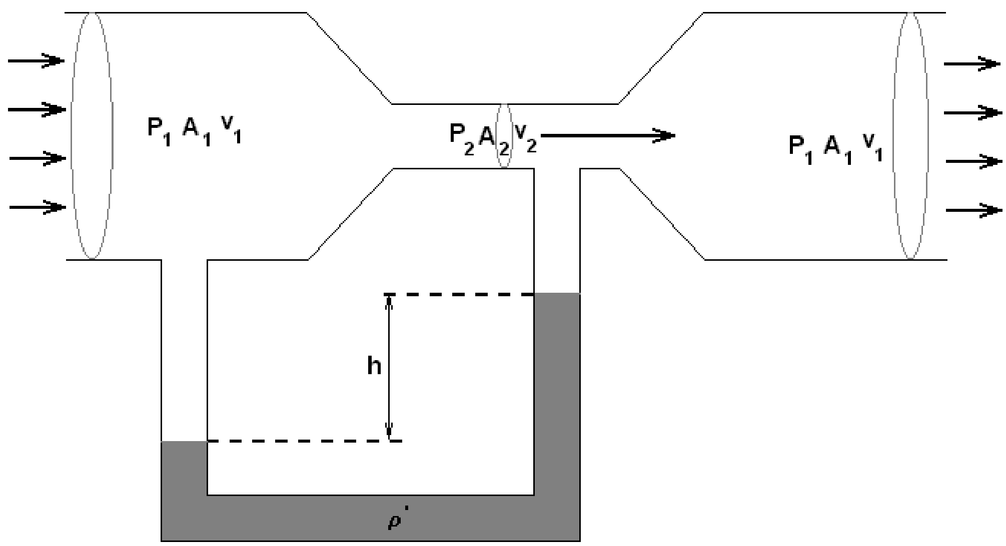
This causes the pressure above the wing to be lower than the pressure below the wing. The pressure difference pushes up on the wing.

This is called lift.

There is a lot more going on with the airplane wing than shown in this diagram. We are ignoring turbulence and the fact that air is a compressible fluid, among other things. But the general idea should be clear. The higher velocity above the wing means less pressure above the wing. The higher pressure below then pushes up on the wing, which keeps the airplane in the air. All planes have a minimum air speed they must maintain in order to fly. It is the speed that creates a pressure difference sufficient to hold the plane in the air. Below that speed there will still be a pressure difference between the air above the wing and the air below the wing. It will simply be too small to support the weight of the plane, and the plane will no longer fly.

<u>Venturi Tube:</u> Variants of this concept can also be used for measuring quantities of gas flowing through a tube, for measuring the speed of subsonic aircraft, for measuring the volumes of oil or natural gas flowing through a pipeline, among others.

For simplicity we will consider here only the velocity of a flowing, non-compressible liquid. When working with gases one has to take adequate account of the fact that the density it not constant, that the gas is compressible, and that the gas temperature changes as the pressure changes. The mathematics for that case is a bit more involved than we want to address here. Therefore, we will only look at the simplest case. Elevation will not be an issue since the center points of our Venturi Tube are at the same elevation everywhere along its length. Even though we envision measuring the pressure difference using a mercury manometer, these are hardly ever used any longer. There are many ingenious mechanical devices used instead for measuring pressure differences.



The centers of all the tubes have a common elevation; therefore the gravity-terms drop out of Bernoulli's equation. This leaves

$$P_1 + ½\rho v_1^2 = P_2 + ½\rho v_2^2 \quad \text{or} \quad P_1 - P_2 = ½\rho v_2^2 - ½\rho v_1^2 = ½\rho (v_2^2 - v_1^2)$$

The pressure difference measured by the mercury manometer is

$$P_1 - P_2 = \rho' gh$$

Assuming we know v_1 and need to find v_2, we begin by assuming an incompressible fluid and use the constant volume flow equation to remove v_1 by substitution. Thus,

$$A_1v_1 = A_2v_2$$

So

$$v_1 = A_2v_2 / A_1$$

Therefore, substituting these results into Bernoulli's equation, yields

$$\rho' gh = \tfrac{1}{2}\rho\,(v_2^2 - (A_2 v_2 / A_1)^2)$$

$$\rho' gh = \tfrac{1}{2}\rho\, v_2^2\,(1 - (A_2 / A_1)^2)$$

This can be solved for v_2^2, as follows,

$$v_2^2 = h \bullet 2g\,(\rho'/\rho) / \{1 - (A_2 / A_1)^2\}$$

Everything to the right of the h consists of constants. The details depend on the sizes of the pipes, the densities of the liquids, and the local value of g. For simplicity, we collect most of these constants into one equipment-constant; call it K_2. Then,

$$v_2^2 = K_2\,(h / \rho) \qquad \text{where } K_2 = 2g\,\rho'/ \{1 - (A_2 / A_1)^2\}$$

If, instead, we know v_2 and need to find v_1, the same method applies. This time,

$$v_1^2 = K_1\,(h / \rho) \qquad \text{where } K_1 = 2g\,\rho' / \{ (A_1 / A_2)^2 - 1\}$$

Since K_1 and K_2 are apparatus specific, it is often easier to simply calibrate the apparatus and find the values of these constants under known conditions rather than trying to calculate them. Proper calibrations mean that only two careful measurements, of K and ρ, are required; not five careful measurements of g, ρ', ρ, A_1, and A_2.