A Mathematical Perspective on Gambling

Molly Maxwell

Abstract. This paper presents some basic topics in probability and statistics, including sample spaces, probabilistic events, expectations, the binomial and normal distributions, the Central Limit Theorem, Bayesian analysis, and statistical hypothesis testing. These topics are applied to gambling games involving dice, cards, and coins.

1. Introduction. In 1654, the well-known gambler the Chevalier de Mere posed a set of problems concerning gambling odds to Blaise Pascal. These problems initiated a series of letters between Pascal and Pierre de Fermat, which established the basis of modern probability theory [**1**, pp. 229–53].

The Chevalier de Mere was an astute gambler who observed a slight advantage in attempting to roll a 6 in four tosses of a single die and a slight disadvantage in attempting to throw two 6s, a result referred to as a "double-6," in twenty-four tosses of two dice [**1**, p. 89]. De Mere noticed that the proportion of the number of tosses to the number of possible outcomes is the same for both games (4:6 for the single-die game, and 24:36 for the double-die game). Thus he concluded that the odds of winning each game should be equal. Yet from his experience of playing these two games, he felt confident that there was an advantage to the single-die game and a disadvantage to the double-die game. Thus he asked Pascal for an explanation for this discrepancy. In their correspondence, Pascal and Fermat laid the groundwork for the concepts of sample space, probabilistic events, expectations, and the binomial distribution.

Section 2 of this paper introduces the ideas of sample spaces and events, and applies these concepts to de Mere's die games, poker, and coin-tossing games. The work in this section confirms de Mere's intuition that the single-die game has a slight advantage ($p = 0.5177$) and the double-die game has a slight disadvantage ($p = 0.4914$). Section 3 defines expectations, and finds the expected payoff for a single trial of the games discussed in Section 2. Section 4 presents the binomial and normal distributions, and explains how these distributions can be used to calculate the odds of making a profit in a large number of trials of these games. Finally, Section 5 develops the concepts of Bayesian analysis and statistical hypothesis testing, and shows how these methods can be used to calculate the likelihood that a given result came from a particular distribution.

2. Sample Spaces and Events. A *sample space* Ω is a set of all possible outcomes of a model of an experiment [**2**, p. 6]. This set is "finest grain"; in other words, all distinct experimental outcomes are listed separately. In addition, the elements in the set are "mutually exclusive"; each trial of the experiment can result in only one of the possible outcomes. The words "outcome" and "event" are commonly interchanged; however, in probability theory these two words have distinct meanings. An *event* is a subset of possible outcomes. As an example, a single roll of a die is an experiment with six possible outcomes, corresponding to the faces that land up. Rolling an even number is an event that includes the experimental outcomes 2, 4, and 6.

The *conditional probability* is the probability of Event A given Event B, and is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

for $P(B) > 0$; see [**2**, pp. 13–16]. Event B is a conditioning event, and redefines the sample space for Event A. This new sample space includes only the outcomes in which B occurs.

Events A and B are called *independent* if the occurrence of one event does not affect the likelihood of the occurrence of the other [**2**, p. 17]. More formally, Events A and B are independent if

$$P(A|B) = P(A) \text{ and } P(B|A) = P(B).$$

The concepts of conditional probability and independence are useful for Bayesian analysis, discussed in Section 5. The idea of independence is also important for the binomial distribution, discussed in Section 4.

We now consider three examples of sample spaces and probabilities. First we discuss die games. A single roll of a die has six possible outcomes, corresponding to the faces that land up. If the die is fair, then the probability of any particular face landing up is 1/6. More formally, the sample space Ω is the set $\{1, 2, 3, 4, 5, 6\}$, and the probability is $P(X = i) = 1/6$. Games that involve rolling two dice have thirty-six possible outcomes since the roll of each die has six possible outcomes. In these games, the dice are distinguishable; therefore, the outcomes (i, j) and (j, i) are counted separately. More formally, the sample space Ω is the set of pairs (i, j) where $i, j \in \{1, 2, 3, 4, 5, 6\}$, and the probability is P(X=i,Y=j) =1/36.

The first game that the Chavalier de Mere discussed with Pascal involves rolling a die four times [**1**, pp. 88–89]. If the die lands on a 6 at least once in four trials, then the gambler wins. If the die never lands on a 6 in the four trials, then the gambler loses. The four tosses are independent. Hence, the probability of not getting a 6 in four trials is $(5/6)^4$, or 0.4823. It follows that the probability of getting at least one 6 in four trials is equal to $1 - 0.4823$, or 0.5177.

The second game discussed by Pascal and de Mere involves simultaneously rolling two dice twenty-four times [**1**, pp. 88–89]. If both dice land on 6, an outcome called a "double-6," then the gambler wins the bet. If a double-6 does not occur, then the gambler loses the bet. As in the first game, the trials are independent. Hence, the probability of not getting a double-6 in twenty-four trials is $(35/36)^{24}$, or 0.5086. It follows that the probability of getting a double-6 in twenty-four trials is equal to $1 - 0.5086$, or 0.4914. De Mere noticed the slight advantage of betting on the first game and the slight disadvantage of betting on the second game. To discern these small diffferences, de Mere must have played these games a large number of times. An estimate for this number is obtained by using statistical hypothesis testing in Section 5.

As a second example of sample spaces and probabilities, we consider card games. A standard deck of cards consists of fifty-two cards, divided into four suits and thirteen ranks [**4**, pp. 47–51]. In the simple case of drawing one card from the deck, the sample space Ω is the fifty-two possible cards, and the probability of getting any particular card is 1/52. Most card games involve hands of cards in which the order of the cards does not matter. The number of possible hands is the binomial coefficent $\binom{n}{x}$, where n is the size of the deck of cards and x is the size of a hand. The *binomial coefficient* $\binom{n}{x}$ is defined by

$$\binom{n}{x}\left(\begin{array}{c} n \\ x \end{array} \right) = \frac{n!}{(n-x)!\,x!}.$$

The symbol $\binom{n}{x}$ is read "n choose x."

In a standard poker game, each player receives a hand of five cards [**4**, p. 51–55]. The sample space Ω for poker is the set of all possible five-card hands, in which order is irrelevant. The size of this sample space is $\binom{52}{5}$, or $2,598,960$. Thus the probability of getting a hand with five particular cards is $1/2,598,960$, or $3.85 \cdot 10^{-7}$. The probability of getting a particular type of hand is computed as follows: Count the number of ways this type of hand can occur, then divide this result by the total number of hands.

For example, a straight flush is a run of five consecutive cards in a particular suit. A straight flush can occur in any of the four suits, and can begin in ten possible positions, 2 through Jack. Thus, there are forty ways to obtain a staight flush. The probability of getting a straight flush is, therefore, $40/2,598,960$, or 0.000015.

As another example, a full house consists of three of a kind and one pair. There are $\binom{13}{1}$, or 13, possible ranks for the three of a kind and $\binom{4}{3}$, or 4, possible combinations of the four cards of this rank. There are $\binom{12}{1}$, or 12, remaining ranks for the pair and $\binom{4}{2}$, or 6, possible combinations of the four cards of this rank. Thus, the probability of getting a full house is

$$\frac{\binom{13}{1} \cdot \binom{4}{3} \cdot \binom{12}{1} \cdot \binom{4}{2}}{\binom{52}{5}} = \frac{3,744}{2,598,960} = 0.001441.$$

Table 2-1 lists possible poker hands, the number of ways in which they can occur, and their probabilities of occurring; it was taken from [**4**, p. 53].

Table 2-1
A table of poker hands and their probabilities

HAND	NUMBER of WAYS	PROBABILITY
Straight flush	40	0.000015
Four of a kind	624	0.000240
Full House	3,744	0.001441
Flush	5,108	0.001965
Straight	10,200	0.003925
Three of a kind	54,912	0.021129
Two pair	123,552	0.047539
One pair	1,098,240	0.422569
Worse than one pair	1,302,540	0.501177
TOTALS	$\binom{52}{5} = 2,598,960$	1.000000

As the last example of sample spaces and probabilities, we discuss coin-tossing games. The sample space Ω for a single toss of a coin consists of two outcomes: heads H, and tails T. If the coin is fair, then the probability of heads and the probability of tails are both $1/2$. The sample space for a game consisting of five tosses of a coin is 2^5, or 32, sequences of the form $X_1X_2X_3X_4X_5$, where each X_i is H with probability 1/2, or T with probability 1/2. The tosses are independent. Therefore, the probability of getting no heads in five tosses is $(1/2)^5$, or 0.03125. Consequently, the probability of getting at least one head in five tosses of a fair coin is $1 - 0.03125$, or 0.96875.

Consider another game that consists of attempting to toss exactly three heads in four trials. The sample space for this game is 2^4, or 16, sequences of the form $X_1X_2X_3X_4$, where each X_i is H with probability 1/2, or T with probability 1/2. The event of getting exactly three heads corresponds to the following sequences: HHHT, HHTH, HTHH, and THHH. Thus, the probability of winning this bet is $4/2^4$, or 0.25. Another expression for this probability is $\binom{4}{3} \cdot (1/2)^3 \cdot (1/2)$, which is interpreted as the number of ways to choose three of the four positions for the heads multiplied by the probability of getting three heads and the probability of getting one tail.

3. Expectations. For an integer random variable X, the *expectation* E(X) is defined to be the sum of all possible experimental values of the variable weighted by their probabilities [**5**, pp. 86–87]. The expectation is written as

$$E(X) = \sum_n n \cdot P(X = n).$$

The expectation of a random variable is its "mean" or "expected value." One application of the concept of expectation is to formalize the idea of the expected payoff of a game. For an integer random variable X with mean $E(X) = \mu$, the *variance* ${\sigma^2}_X$ is defined by the formula,

$$\sigma_X^2 = E[(X - \mu)^2],$$

see [**3**, p. 119]. The variance can also be expressed as ${\sigma^2}_X = E(X^2) - \mu^2$; this equation is proved in [**3**, p. 119]. The variance measures the spread of the distribution. The *standard deviation* σ_X is the positive square root of the variance. This concept is used in Sections 4 and 5.

We now compute the expected payoffs of the games discussed in Section 2. First we find the expected payoffs for die games. Suppose a gambler gains one dollar for winning a game, and gives up one dollar for losing. In de Mere's single-die game, the expected payoff is

$$E(X) = \$1 \cdot (0.5177) + (-1\$) \cdot (0.4823) = \$0.0354,$$

or approximately four cents. In his double-die game, the expected payoff is

$$E(X) = \$1 \cdot (0.4914) + (-1\$) \cdot (0.5086) = -\$0.0172,$$

or approximately a two-cent loss.

Next we compute the expected payoff for casino poker. Suppose a gambler wages a one-dollar bet on a five-card hand. The gambler's payoff depends on the type of hand, with a rare hand such as a straight flush resulting in a higher payoff than a more common hand such as one pair. Table 3-1 provides one possible payoff scheme. According to this scheme, the expected earnings for one game are

$$\begin{aligned} E(X) &= \$1000 \cdot (0.000015) + \$500 \cdot (0.000240) + \$50 \cdot (0.001441) \\ &\quad + \$25 \cdot (0.001965) + \$10 \cdot (0.03925) + \$5 \cdot (0.021129) \\ &\quad + \$1 \cdot (0.047539) + \$0 \cdot (0.422569) + (-\$1) \cdot (0.501177) \\ &= -\$0.052568, \end{aligned}$$

or approximately a five-cent loss.

Table 3-1
A table of poker hands and their payoffs

HAND	PROBABILITY	PAYOFF
Straight flush	0.000015	\$1000
Four of a kind	0.000240	\$500
Full House	0.001441	\$50
Flush	0.001965	\$25
Straight	0.003925	\$10
Three of a kind	0.021129	\$5
Two pair	0.047539	\$1
One pair	0.422569	\$0
Worse than one pair	0.501177	−\$1
TOTAL	1.000000	

Finally, we find the expected payoffs for coin-tossing games. Suppose a gambler wins one dollar for tossing heads and loses one dollar for tossing tails with a fair coin. His expected payoff is

$$E(X) = \$1 \cdot (0.5) + (-1\$) \cdot (0.5) = 0.$$

Suppose he bets one dollar on getting exactly three heads in four tosses. Then his expected payoff is

$$E(X) = \$1 \cdot (0.25) + (-1\$) \cdot (0.75) = -\$0.50,$$

or a fifty-cent loss.

4. Repeated Trials. The previous section explored a gambler's expected earnings on a single trial of a game. A gambler may, however, want to calculate his expected earnings if he plays the game many times. This section presents methods for calculating the expected payoff and estimating the chances of making a profit on many repetitions of a game.

Consider the problem of calculating the probability of winning t trials out of n, when the probability of winning each trial is set [**2**, pp. 124–26]. The n trials can be expressed as independent, identically distributed random variables $X_1, \cdots, X_n$. Each X_i has two outcomes: 1, which represents a winning trial, and occurs with probability p, and 0, which represents a losing trial, and occurs with probability $1 - p$. The expectation for each X_i is

$$E(X_i) = 1 \cdot p + 0 \cdot (1 - p) = p.$$

The random variable S_n represents the number of winning trials in a series of n trials, and is given by

$$S_n = X_1 + X_2 + \cdots + X_n.$$

Therefore, the expected number of winning trials is

$$\begin{aligned} E(S_n) &= E(X_1 + X_2 + \cdots + X_n) \\ &= E(X_1) + E(X_2) + \cdots + E(X_n) \\ &= n \cdot E(X_i) = n \cdot p, \end{aligned}$$

since the random variables X_i are independent and identically distributed.

Furthermore, the probability of winning *exactly* t trials out of n can be calculated by multiplying the number $\binom{n}{t}$ of ways to choose t of the n trials to be successes by the probability p^t of winning t games, and the probability $(1-p)^{n-t}$ of losing $n-t$ games. This is the *binomial distribution*, and is defined by

$$P(S_n = t) = \binom{n}{t} \cdot p^t \cdot (1-p)^{n-t},$$

for $0 \le t \le n$. As calculated above, the expectation of S_n is

$$E(S_n) = n \cdot p.$$

Moreover, the variance of S_n is

$$\sigma^2{}_{S_n} = n \cdot p \cdot (1-p).$$

The probability of winning *at least* t games out of n is the sum of the probabilities of winning t through n games since these events are mutually exclusive. This probability is denoted by

$$P(S_n \ge t) = \sum_{i=t}^{n} \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i},$$

for $0 \le t \le n$.

As an example, consider de Mere's single-die game, in which the probability of success on each trial is 0.5177. In thirty trials, the expected number of successes is

$$E(S_{30}) = 30 \cdot (0.5177) = 15.5,$$

which rounds to 16. The probability of winning exactly nineteen trials out of thirty is

$$P(S_{30} = 19) = \binom{30}{19} \cdot (0.5177)^{19} \cdot (0.4823)^{11} = 0.0663.$$

Furthermore, the probability of winning at least nineteen trials out of thirty is

$$P(S_{30} \ge 19) = \sum_{i=19}^{30} \binom{30}{19} \cdot (0.5177)^i \cdot (0.4823)^{30-i} = 0.1388.$$

The normal, or Gaussian, distribution is important because it describes many probabilistic phenomena [**2**, pp. 207–11]. Consider a normal random variable w with mean μ and variance σ^2. The *normal distribution* is defined by

$$P(w \le a) = \int_{-\infty}^{a} \frac{1}{\sigma\sqrt{2\pi}} e^{-((w-\mu)/\sigma)^2/2} dw.$$

A standard normal random variable z has $\mu = 0$ and variance $\sigma^2 = 1$. The *standard normal distribution* is defined by

$$P(z \leq a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \Phi(a).$$

Most probability and statistics textbooks provide tables for $\Phi(x)$; for example, see [**3**, pp. 676–77] and [**5**, p. 177].

The Central Limit Theorem states that the probability distribution for a sum S_n of independent, identically distributed random variables X_i approaches the normal distribution as the number of random variables goes to infinity [**2**, pp. 215–19]. This result is remarkable as it does not depend on the distribution of the random variables. The Central Limit Theorem states that

$$\lim_{n\to\infty} P(S_n \leq t) = \Phi\left(\frac{t - E(S_n)}{\sigma_{S_n}}\right).$$

Hence, for finite, but large values of n, the following approximation holds:

$$P(S_n \leq t) \approx \Phi\left(\frac{t - E(S_n)}{\sigma_{S_n}}\right).$$

For example, in de Mere's single-die game, the random variable S_{30} represents the number of winning games out of thirty. The random variable

$$z = \frac{S_{30} - E(S_{30})}{\sigma_{S_{30}}}$$

is approximately standard normal. Therefore, the probability of winning at least nineteen trials out of thirty is

$$\begin{aligned} P(S_{30} \geq 19) &= 1 - P(S_{30} \leq 19) \\ &= 1 - P\left(z \leq \left(\frac{19 - E(S_n)}{\sigma_{S_n}}\right)\right) \\ &\approx 1 - \Phi\left(\frac{19 - (30 \cdot 0.5177)}{\sqrt{30 \cdot 0.5177 \cdot 0.4823}}\right) \\ &= 1 - 0.8577 \\ &= 0.1423. \end{aligned}$$

This probability provides a close estimate of the actual value (0.1388), which was found using the binomial distribution. Most probability and statistics textbooks provide graphs of the binomial and normal distributions for various values of n; for example, see [**2**, p. 217] and [**3**, p. 272]. The graphs indicate that as $n \to \infty$, the normal distribution forms a good approximation for the binomial distribution.

5. Bayesian Analysis and Statistical Hypothesis Testing. So far we have looked only at the probabilities of observing particular outcomes and the expected payoffs for various games with known distributions. This last section focuses on estimating the likelihood that an observed outcome came from a particular distribution. For example, a gambler tosses forty-one heads in one-hundred tosses of a coin, and wishes to know if the coin is fair. Bayesian analysis and statistical hypothesis testing are two methods for determining the likelihood that an outcome came from a particular distribution.

Bayesian analysis tests the likehlihood that an observed outcome came from a particular probability distribution by treating unknown parameters of the distribution as random variables [**2**, pp. 250–57]. These unknown parameters have their own probability distributions, which often are derived from the assumed probability distribution of the outcome. The derivations use Bayes' Theorem, which states that

$$P(A_i|B) = \frac{P(B|A_i) \cdot P(A_i)}{\sum_{i=1}^{N} P(B|A_i) \cdot P(A_i)}.$$

For example, let θ be an unknown parameter that has two possible values, corresponding to the events $A_1 = \{\theta = \theta_1\}$ and $A_2 = \{\theta = \theta_2\}$. The *a priori* probabilities for A_1 and A_2 are $P(A_1)$ and $P(A_2)$. They do not take the information gained from the occurrence of event B into account. Event B is an observed outcome, which has an assumed probability ditribution that depends on the unknown parameter θ. Consider the event $A_1 = \{\theta = \theta_1\}$. By Bayes' Theorem, the probability of A_1 given B is

$$P(A_1|B) = \frac{P(B|A_1) \cdot P(A_1)}{P(B|A_1) \cdot P(A_1) + P(B|A_2) \cdot P(A_2)}.$$

Bayesian analysis can be applied to the example of the gambler who tossed forty-one heads in one-hundred coin tosses, and wishes to know if his coin is fair. Assume that the gambler has two coins, a fair coin ($p = 1/2$) and a biased coin ($p = 1/4$). Event A_1 corresponds to his selecting the fair coin, and Event A_2 corresponds to his selecting the biased coin. He randomly selects the coin; therefore, the *a priori* probabilites, $P(A_1)$ and $P(A_2)$, are both equal to 1/2. Event B is the observed outcome that $S_{100} = 41$. By Bayes' Theorem, the probability that the gambler used the fair coin is

$$\begin{aligned} P\Big(p = \frac{1}{2} \;\Big|\; S_{100} = 41\Big) &= \frac{P\left(S_{100} = 41 \mid p = \frac{1}{2}\right) \cdot P\left(p = \frac{1}{2}\right)}{P\left(S_{100} = 41 \mid p = \frac{1}{2}\right) \cdot P\left(p = \frac{1}{2}\right) + P\left(S_{100} = 41 \mid p = \frac{1}{4}\right) \cdot P\left(p = \frac{1}{4}\right)} \\ &= \frac{\binom{100}{41} \cdot \left(\frac{1}{2}\right)^{41} \cdot \left(\frac{1}{2}\right)^{59} \cdot \left(\frac{1}{2}\right)}{\binom{100}{41} \cdot \left(\frac{1}{2}\right)^{41} \cdot \left(\frac{1}{2}\right)^{59} \cdot \left(\frac{1}{2}\right) + \binom{100}{41} \cdot \left(\frac{1}{4}\right)^{41} \cdot \left(\frac{3}{4}\right)^{59} \cdot \left(\frac{1}{2}\right)} \\ &= 0.989. \end{aligned}$$

Bayesian analysis combines prior knowledge of the unknown parameter (in this case, the *a priori* probabilities $P(p = 1/2)$ and $P(p = 1/4)$ with experimental evidence to form a better estimate of this parameter.

In statistical hypothesis testing, an observed outcome is compared to an assumed model to determine the likelihood that the outcome came from the model [**5**, pp. 163–65]. If the outcome deviates significantly from the expected outcome of the assumed model, then it is unlikely that the result came from this model. Statistical hypothesis testing, therefore, involves determing what level of deviation from the expected outcome is significant.

Consider a random variable Y that has a probability distribution that relies on the parameter θ. This method tests a null hypothesis $H_0 : \theta = \theta_0$ against an alternative hypothesis H_1, which may take the form $\theta < \theta_0$, $\theta > \theta_0$, or $\theta \neq \theta_0$; see [**3**, pp. 336–39]. A significance level α is chosen for the test, and a critical value c that depends on α is calculated. The critical region consists of the critical value, and all values that are more extreme in the direction of the alternative hypothesis. If the observed outcome of Y falls within the critical region, then H_0 is rejected and H_1 is accepted. However, if the observed outcome of Y does not fall within the critical region, then H_0 is accepted.

The coin-tossing example can be worked out using statistical hypothesis testing. The hypothesis $H_0 : p = 1/2$ is tested against an alternative hypothesis $H_1 : p \neq 1/2$. In the model where H_0 is true, the expectation is $E(S_{100}) = 100 \cdot 1/2 = 50$, and the variance is ${\sigma^2}_{S_{100}} = 100 \cdot 1/2 \cdot 1/2 = 25$. By the Central Limit Theorem, the variable S_{100} has an almost normal distribution. Therefore, the variable $z = \frac{S_{100}-50}{5}$ has an approximately standard normal distribution. Thus,

$$P\Big(-1.96 \leq \frac{S_{100} - 50}{5} \leq 1.96\Big) = 0.95, \text{ or } P\left(40.2 \leq S_{100} \leq 59.8\right) = 0.95.$$

Since the observed outcome, $S_{100} = 41$, falls within this interval, H_0 is accepted at the $\alpha = 0.05$ significance level. The *significance level* is the probability that the assumed model H_0 is erroneously rejected. This error is commonly referred to as a type I error. Likewise, a type II error is the event that the assumed model H_0 is erroneously accepted. The probability of this type of error is denoted as β. Ideally, we want α and β to be as small as possible; however, decreasing both involves increasing the sample size.

For example, when $H_0 : p = 1/2$ is tested against $H_1 : p = 1/4$ in a coin-tossing model, H_0 will be rejected if the number of heads S_{100} is below some value c, known as the *critical value*. In this case, α represents the probability that S_{100} is less than c when $p = 1/2$, and β represents the probability that S_{100} is greater than c when $p = 1/4$. For instance, if $\alpha = 0.001$ and $\beta = 0.001$, this critical value c and the sample size n can be found by solving the equations for α and β. Namely,

$$\alpha = 0.001 = P\Big(S \leq c \;\Big|\; p = \frac{1}{2}\Big) = P\bigg(\frac{S - \frac{n}{2}}{\sqrt{\frac{n}{4}}} \leq \frac{c - \frac{n}{2}}{\sqrt{\frac{n}{4}}}\bigg) = P(z \leq z_{0.001}).$$

Because $z_{0.001} = -3.090$, it follows that

$$\frac{c - \frac{n}{2}}{\sqrt{\frac{n}{4}}} \leq -3.090, \text{ and } c \approx -3.090 \cdot \sqrt{\frac{n}{4}} + \frac{n}{2}. \qquad (5-1)$$

Likewise,

$$\beta = 0.001 = P\Big(S \geq c \;\Big|\; p = \frac{1}{4}\Big) = P\left(\frac{S - \frac{n}{4}}{\sqrt{n \cdot \frac{1}{4} \cdot \frac{3}{4}}} \geq \frac{c - \frac{n}{4}}{\sqrt{n \cdot \frac{1}{4} \cdot \frac{3}{4}}}\right) = P(z \geq z_{0.001}).$$

Because $z_{0.001} = 3.090$, it follows that

$$\frac{c - \frac{n}{4}}{\sqrt{n \cdot \frac{1}{4} \cdot \frac{3}{4}}} \geq 3.090, \text{ and } c \approx 3.090 \cdot \sqrt{n \cdot \frac{1}{4} \cdot \frac{3}{4}} + \frac{n}{4}. \qquad (5-2)$$

Combining Expressions (5-1) and (5-2) yields that

$$-3.090 \cdot \sqrt{\frac{n}{4}} + \frac{n}{2} \approx 3.090 \cdot \sqrt{n \cdot \frac{1}{4} \cdot \frac{3}{4}} + \frac{n}{4}.$$

Thus, n is equal to 132.8, which is rounded to 133, and c is equal to 48.6, which is rounded to 49. To reduce the probabilities of Type I and Type II errors, α and β, to 0.001, the number of trials must be increased to 133.

As mentioned in the Introduction, the Chevalier de Mere noticed that his bet on the double-die game was less than chance [**1**, pp. 183–84]. Some of the techniques in this last section can provide an estimate for the number of trials de Mere must have played to legitimately claim that the probability of success was indeed less than 1/2. Assume that de Mere played the game a large number of times, and found that he rolled a double-6 in 49% of the games. To determine whether he could legitimately accept that $p < 1/2$, de Mere would need to test the hypothesis $H_0 : p = 1/2$ against the hypothesis $H_1 : p < 1/2$. Let Y be the number of times de Mere rolled a double-6 in n trials. From the Central Limit Theorem, it follows that Y has an almost normal distribution. Thus, the variable

$$z = \frac{Y - np}{\sqrt{n \cdot p \cdot (1-p)}} = \frac{\frac{Y}{n} - p}{\sqrt{\frac{p \cdot (1-p)}{n}}}$$

has an approximate standard normal distribution. For a 0.05 significance level, it follows that

$$P\left(\frac{\frac{Y}{n} - \frac{1}{2}}{\sqrt{\frac{\frac{1}{2} \cdot \frac{1}{2}}{n}}} \leq z_{0.05}\right) = 0.05 = \alpha.$$

If de Mere estimated that $\frac{Y}{n}$ is $\frac{49}{100}$, then

$$\frac{\frac{49}{100} - \frac{1}{2}}{\sqrt{\frac{1}{4n}}} \leq z_{0.05} = -1.645.$$

Consequently,

$$n = 6765.$$

To legitimately accept the hypothesis $H_1 : p < 1/2$ for the double-die game, de Mere must have played 6765 games.

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