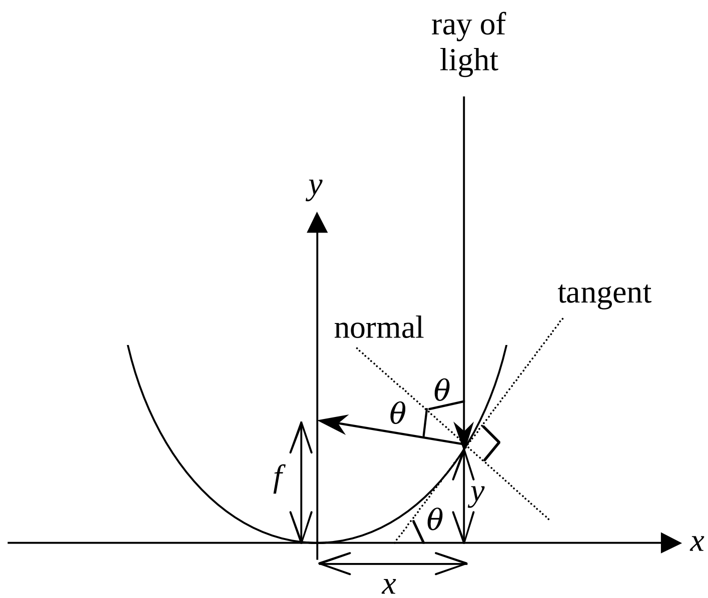
Parabolic Mirror—C.E. Mungan, Fall 1999

Prove that a parabolic mirror brings collimated incident light to a single focal point. In contrast, determine how much aberration is produced in the focal plane of a spherical mirror.



Let the shape of the parabolic mirror be defined by $y = mx^2$. Then the slope of the tangent line can be expressed either as the derivative or in terms of the angle of incidence θ, so that $\tan\theta = 2mx$. But from the above diagram it is clear that the focal length is

$$f = y + \frac{x}{\tan 2\theta}.$$

Thus, by using the trig identity

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{4mx}{1-4m^2x^2},$$

one finds that

$$f = mx^2 + \frac{1-4m^2x^2}{4m} = \frac{1}{4m}.$$

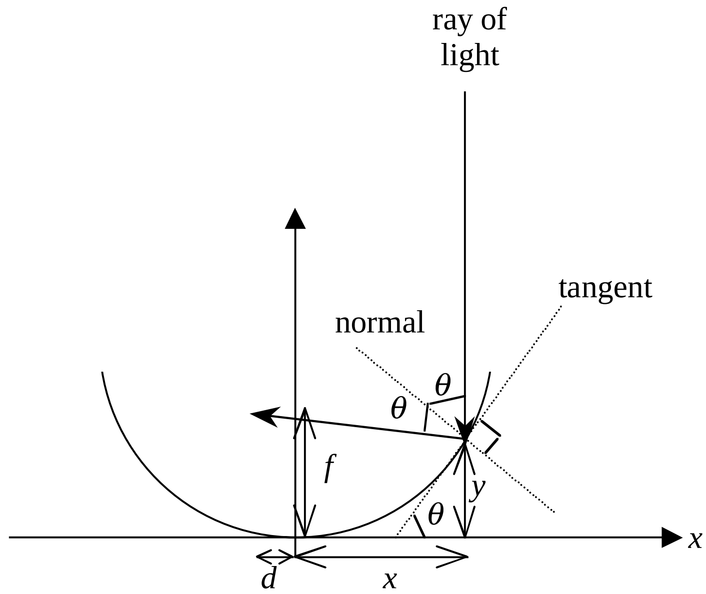
This is independent of x, thus proving the assertion. It also gives a useful relation between the focal length and shape of a parabolic mirror.

Incidentally, the line $y = -f$ is called the directrix. It has the property that it is just as far from any point on the parabola as the focus is. That is

$$\sqrt{(f-y)^2 + x^2} = y + f,$$

as follows by squaring both sides and substituting $y = x^2/4f$.

Now, a spherical mirror has a stronger curvature than a parabolic mirror of the same focal length. (After all, it curves all the way around on itself.) Thus the tangent has a steeper slope and our fiducial light ray crosses the focal plane to the left of the principal axis, as sketched below.



The equation for a circle of radius r with the origin at its bottom edge is $x^2 + (y-r)^2 = r^2$, so that

$$y = r - \sqrt{r^2 - x^2}.$$

To find f, we expand this to lowest order in x and match its curvature to that of a parabola,

$$y \to \frac{x^2}{2r} \equiv \frac{x^2}{4f} \Rightarrow f = \tfrac{1}{2}r$$

which is a well-known result. Next, we again set the derivative of the shape to the tangent of the angle of incidence,

$$\tan\theta = \frac{dy}{dx} = \frac{x}{\sqrt{r^2 - x^2}}.$$

From the diagram we see that the deviation $d = (f-y)\tan 2\theta - x$. Using the same double-angle trig identity as before and inserting the above three set-off equations leads to the final result

$$d = x\frac{1-\sqrt{1-(x/r)^2}}{1-2(x/r)^2}.$$

More practically, suppose that an incident beam of diameter $D \equiv 2x$ centered about and parallel to the principal axis illuminates the mirror. Then the diameter of the blur spot will be

$$2d \cong \frac{D^3}{8r^2},$$

where I assumed that D is much smaller than the diameter of curvature of the mirror. For example, if $D = f$ (which is probably tighter of a focus than one would ever use and implies that D is 25% of the diameter of curvature) then the blur spot diameter is 3.6% of D according to the exact formula and 3.1% of D according to this approximation.