Chapter 7. Response of First-Order RL and RC Circuits

7.3 The Step Response of RL and RC Circuits

Finding the currents and voltages in first-order RL or RC circuits when either dc voltage or current sources are suddenly applied.

The Step Response of an RL Circuit

The circuit is shown in Fig. 7.16.
Energy stored in the inductor at the time the switch is closed is given in terms of a **nonzero** initial current $i(0)$.

The task is to find the expressions for the current in the circuit and for the voltage across the inductor after the switch has been closed.

We derive the differential equation that describes the circuit and we solve the equation.

After the switch has been closed, Kirchhoff's voltage law requires that

$$V_S = Ri + L\frac{di}{dt} \tag{7.29}$$

which can be solved for the current by separating the variables i and t, then integrating.
The first step is to solve Eq. (7.29) for di/dt:

$$\frac{di}{dt} = \frac{-Ri + V_S}{L} = \frac{-R}{L}\left(i - \frac{V_S}{R}\right) \tag{7.30}$$

Next, we multiply both sides by dt.

$$di = \frac{-R}{L}\left(i - \frac{V_S}{R}\right)dt \tag{7.31}$$

We now separate the variables in Eq. (7.32) to get

$$\frac{di}{i - (V_S / R)} = \frac{-R}{L}dt \tag{7.32}$$

and then integrate both sides. Using x and y as variables for the integration, we obtain

$$\int_{I_0}^{i(t)} \frac{dx}{x-(V_S/R)} = \frac{-R}{L}\int_0^t dy \tag{7.33}$$

where I_0 is the current at $t=0$ and $i(t)$ is the current at any $t>0$.
Therefore

$$\ln\frac{i(t)-(V_S/R)}{I_0-(V_S/R)} = \frac{-R}{L}t \tag{7.34}$$

from which

$$\frac{i(t)-(V_S/R)}{I_0-(V_S/R)} = e^{-(R/L)t}$$

or

$$i(t) = \frac{V_S}{R} + \left(I_0 - \frac{V_S}{R}\right)e^{-(R/L)t} \tag{7.35}$$

When the initial energy in the inductor is zero, I_0 is zero. Thus eq. (7.35) reduces to

$$i(t) = \frac{V_S}{R} - \frac{V_S}{R} e^{-(R/L)t} \tag{7.36}$$

Eq. (7.36) indicates that after the switch has been closed, **the current increases exponentially** from zero to a **final value** V_S / R.

The **time constant** of the circuit, L/R, determines the **rate of increase**. One time constant after the switch has been closed, the current will have reached approximately 63% of its final value, or

$$i(\tau) = \frac{V_S}{R} - \frac{V_S}{R} e^{-1} \approx 0.6321 \frac{V_S}{R} \tag{7.37}$$

If the current were to continue to increase at its initial rate, it would reach its final value at $t = \tau$; that is because

$$\frac{di}{dt} = \frac{-V_S}{R} \left(\frac{-1}{\tau} \right) e^{-t/\tau} = \frac{V_S}{L} e^{-t/\tau} \tag{7.38}$$

the initial rate at which $i(t)$ increases is

$$\frac{di}{dt}(0) = \frac{V_S}{L} \tag{7.39}$$

If the current were to continue to increase at this rate, the expression for i would be

$$i = \frac{V_S}{L} t \tag{7.40}$$

from which, at $t = \tau$,

$$i = \frac{V_S}{L} \cdot \frac{L}{R} = \frac{V_S}{R} \tag{7.41}$$

Equations (7.36) and (7.40) are plotted in Fig. 7.17. The values given by Eqs. (7.37) and (7.41) are also shown in this figure.

The **voltage** across an inductor is $L\,di/dt$, so from Eq. (7.35), for $t \geq 0^+$,

$$v = L\left(\frac{-R}{L}\right)\left(I_0 - \frac{V_S}{R}\right)e^{-(R/L)t} = \left(V_S - I_0 R\right)e^{-(R/L)t} \tag{7.42}$$

The voltage across the inductor is zero before the switch is closed. Eq. (7.42) indicates that the inductor voltage jumps to $V_S - I_0 R$ at the instant the switch is closed and then **decays exponentially** to zero.

Does the value of v at $t = 0^+$ makes sense?

Because the initial **current** is I_0 and the inductor prevents **an instantaneous change in current**, the current is I_0 in the instant after the switch has been closed.
The **voltage** drop across the resistor is $I_0 R$, and the voltage impressed across the inductor is the source voltage minus the voltage drop, that is, $V_S - I_0 R$.
When the initial inductor current is zero, Eq. (7.42) simplifies to

$$v = V_S e^{-(R/L)t} \tag{7.43}$$

If the initial current is zero, the voltage across the inductor jumps to V_S. We also expect the inductor voltage to approach zero as t increases, because the current in the circuit is approaching the constant value of V_S / R.

Fig. 7.18 shows the plot of Eq. (7.43) and the relationship between the time constant and the initial rate at which the inductor voltage is decreasing.

If there is an initial current in the inductor, Eq. (7.35) gives the solution for it. The algebraic sign of I_0 is positive if the initial current is in the same direction as i; otherwise, I_0 carries a negative sign.

Example 7.5

The switch shown in Fig. 7.19 has been in position a long time. At $t = 0$, the switch moves from a to b. The switch is a make-before-break type; so, there is no interruption of current through the inductor.

a) Find the expression $i(t)$ for $t \geq 0$
b) What is the initial voltage across the inductor just after the switch has been moved to position b?
c) Does the initial voltage make sense in terms of circuit behavior?
d) How many milliseconds after the switch has been moved does the inductor voltage equal 24 V?
e) Plot both $i(t)$ and $v(t)$ versus t.

- We can also describe the **voltage** $v(t)$ across the inductor directly, not just in terms of the circuit current.

We begin by noting that the voltage across the resistor is the difference between the source voltage and the inductor voltage. We write

$$i(t) = \frac{V_S}{R} - \frac{v(t)}{R} \qquad (7.44)$$

where V_S is a constant. Differentiating both sides with respect to time yields

$$\frac{di}{dt} = -\frac{1}{R} \cdot \frac{dv}{dt} \qquad (7.45)$$

Multiply each side of Eq. (7.45) by the inductance L.

$$v = -\frac{L}{R} \cdot \frac{dv}{dt} \qquad (7.46)$$

Putting Eq. (7.46) into standard form yields

$$\frac{dv}{dt} + \frac{R}{L}v = 0 \qquad (7.47)$$

Verify that the solution to Eq. (7.47) is identical to that given in Eq. (7.42).

$$v = L\left(\frac{-R}{L}\right)\left(I_0 - \frac{V_S}{R}\right)e^{-(R/L)t} = \left(V_S - I_0 R\right)e^{-(R/L)t} \qquad (7.42)$$

At this point, a **general observation** about the step response of an RL circuit is pertinent.

When we derived the differential equation for the inductor current, we obtained Eq. (7.29). We now rewrite Eq. (7.29) $V_S = Ri + L\frac{di}{dt}$ as

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V_S}{L} \qquad (7.48)$$

Observe that Eqs. (7.47) and (7.48) have the same form. Specifically, **each equates the sum of the first derivative of the variable and a constant times the variable to a constant value.**

In (7.47), the constant on the right-hand side is zero; hence this equation takes on the same form as the **natural response** equations.

In both (7.47) and (7.48), the constant multiplying the dependent variable is the reciprocal of the time constant, that is, $\frac{R}{L} = \frac{1}{\tau}$.

We encounter a similar situation in the derivations for the step response of an RC circuit.

The Step Response of an RC Circuit

We can find the step response of a first-order RC circuit by analyzing the circuit shown in Fig. 7.21.

For mathematical convenience, we choose the Norton equivalent of the network connected to the equivalent capacitor. Summing the currents away from the top node in Fig. 7.21 generates the differential equation

$$C\frac{dv_C}{dt}+\frac{v_C}{R}=I_S \qquad (7.49)$$

Division by C gives

$$\frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{I_S}{C} \tag{7.50}$$

Comparing Eq. (7.50) with Eq. (7.48)

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V_S}{L} \tag{7.48}$$

reveals that the form of the solution for v_C is the same as that for the current in the inductive circuit, namely, Eq. (7.35).

$$i(t) = \frac{V_S}{R} + \left(I_0 - \frac{V_S}{R}\right)e^{-(R/L)t} \tag{7.35}$$

Therefore, by simply substituting the appropriate variables and coefficients, we can write the solution for v_C directly.

The translation requires that

I_S **replace** V_S

C **replace** L

$1/R$ **replace** R

V_0 **replace** I_0**.**

We get

$$v_C = I_S R + (V_0 - I_S R)e^{-t/RC}, \quad t \geq 0 \tag{7.51}$$

A similar derivation for the current in the capacitor yields the differential equation

$$\frac{di}{dt} + \frac{1}{RC} i = 0 \tag{7.52}$$

Eq. (7.52) has the same form as Eq. (7.47)

$$\frac{dv}{dt} + \frac{R}{L} v = 0 \tag{7.47}$$

hence the solution for i is obtained by using the same translations used for the solution of Eq. (7.50). Thus

$$i = \left(I_S - \frac{V_0}{R} \right) e^{-t/RC}, \quad t \geq 0^+ \tag{7.53}$$

where V_0 is the initial value for v_C, the voltage across the capacitor.

Let's see if the solutions for the RC circuit make sense in terms of known circuit behavior.

From Eq. (7.51), note that the initial voltage across the capacitor is V_0, the final voltage across the capacitor is $I_S R$, and the time constant of the circuit is RC.

Also note that the solution for v_C is valid for $t \geq 0$. These observations are consistent with the behavior of a capacitor in parallel with a resistor when driven by a constant current source.
Equation (7.53) predicts that the current in the capacitor at $t = 0^+$ is $I_S - V_0/R$.
This prediction makes sense because the capacitor voltage cannot change instantaneously, and therefore the initial current in the resistor is V_0/R. The capacitor branch current changes instantaneously from zero at $t = 0^-$ to $I_S - V_0/R$ at $t = 0^+$. The capacitor current is zero at $t = \infty$. Also note that the final value of $v = I_S R$.

Example 7.6

The switch in the circuit shown in Fig. 7.22 has been in position 1 for a long time. At $t = 0$, the switch moves to position 2. Find

a) $v_0(t)$ for $t \geq 0$

b) $i_0(t)$ for $t \geq 0^+$

7.4 A General Solution for Step and Natural Response

The general approach to finding either the natural response of the step response of the first-order RL and RC circuits shown in Fig. 7.24 is based on their differential equations being the same.
To generalize the solution of these four possible circuits, we let $x(t)$ represent the unknown quantity, giving $x(t)$ four possible values.

It can represent the current or voltage at the terminals of an inductor or the current or voltage at the terminals of a capacitor.
From the previous eqs. (7.47), (7.48), (7.50), and (7.52), we know that the differential equation describing any one of the four circuits in Fig. (7.24) takes the form

$$\frac{dx}{xt} + \frac{x}{\tau} = K \qquad (7.54)$$

where the value of the constant K can be zero.

Because the sources in the circuit are constant voltages and/or currents, the final value of x will be constant; that is, the final value must satisfy (7.54), and, when x reaches its final value, the derivative dx/dt must be zero.
Hence

$$x_f = K\tau \tag{7.55}$$

where x_f represents the final value of the variable.
We solve (7.54) by separating the variables, beginning by solving for the first derivative:

$$\frac{dx}{dt} = \frac{-x}{\tau} + K = \frac{-(x - K\tau)}{\tau} = \frac{-(x - x_f)}{\tau} \tag{7.56}$$

In writing (7.56), we used (7.55) to substitute x_f for $K\tau$. We now multiply both sides of (7.56) by dt and divide by $x - x_f$ to obtain

$$\frac{dx}{x - x_f} = \frac{-1}{\tau} dt \tag{7.57}$$

Integrate (7.57). To obtain as general a solution as possible, we use time t_0 as the lower limit and t as the upper limit.

Time t_0 corresponds to the time of the switching or other change. Previously we assumed that $t_0 = 0$, but this change allows the switching to take place at any time. Using u and v as symbols of integration, we get

$$\int_{x(t_0)}^{x(t)} \frac{du}{u - x_f} = -\frac{1}{\tau} \int_{t_0}^{t} dv \qquad (7.58)$$

Carrying out the integration called for in (7.58) gives

$$x(t) = x_f + \left[x(t_0) - X_f\right] e^{-(t-t_0)/\tau} \qquad (7.59)$$

The significance of this equation is

$$\begin{matrix}\text{the unknown} \\ \text{variable as a} \\ \text{function of time}\end{matrix} = \begin{matrix}\text{the final} \\ \text{value of the variable}\end{matrix}$$

$$+\begin{bmatrix}\text{the initial} & & \text{the final} \\ \text{value of the} & - & \text{value of the} \\ \text{variable} & & \text{variable}\end{bmatrix} \times e^{\frac{-[t - \text{time of switching}]}{\text{time constant}}} \qquad (7.60)$$

In many cases, the time of switching - t_0- is zero.

When computing the step and natural response of circuits, follow these steps:

1. **Identify the variable of interest for the circuit. For RC circuits, it is most convenient to choose the capacitive voltage; for RL circuits, it is best to choose the inductive current.**

2. **Determine the initial value of the variable, which is its value at t_0. Note that if we choose capacitive voltage or inductive current as variable of interest, it is not necessary to distinguish between $t = t_0^-$ and $t = t_0^+$. This is because they both are continuous variables. If we choose another variable, we need to remember that its initial value is defined at $t = t_0^+$.**

3. **Calculate the final value of the variable, which is the value as $t \to \infty$.**

4. **Calculate the time constant for the circuit.**

With these quantities we can use Eq. (7.60) to produce an equation describing the variable of interest as a function of time.