MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

8.02

Review C: Work and Kinetic Energy

Work and Kinetic Energy

C.1 Energy

C.1.1 The Concept of Energy

The concept of energy helps us describe many processes in the world around us.

- Falling water releases stored "gravitational potential energy" turning into a "kinetic energy" of motion. This "mechanical energy" can be used to spin turbines and alternators doing "work" to generate electrical energy. It's sent to you along power lines. When you use any electrical device such as a refrigerator, the electrical energy turns into mechanical energy to make the refrigerant flow to remove 'heat' (the kinetic motion of atoms), from the inside to the outside.

- "Human beings transform the stored chemical energy of food into various forms necessary for the maintenance of the functions of the various organ system, tissues and cells in the body."[1] This "catabolic energy" is used by the human to do work on the surroundings (for example pedaling a bicycle) and release heat.

- Burning gasoline in car engines converts "chemical energy" stored in the atomic bonds of the constituent atoms of gasoline into heat that then drives a piston. With gearing and road friction, this motion is converted into the movement of the automobile.

- Stretching or compressing a spring stores 'elastic potential energy' that can be released as kinetic energy.

- The process of vision begins with stored "atomic energy" released as electromagnetic radiation (light) that is detected by exciting atoms in the eye, creating chemical energy.

- When a proton fuses with deuterium, (deuterium is a hydrogen atom that has an extra neutron along with the proton in the nucleus), helium three is formed (two protons and one neutron) along with radiant energy in the form of photons. The mass of the proton and deuterium are greater than the mass of the helium. This "mass energy" is carried away by the photon.

These energy transformations are going on all the time in the manmade world and the natural world involving different forms of energy: kinetic energy, gravitational energy, heat energy, elastic energy, electrical energy, chemical energy, electromagnetic energy,

[1] George B. Benedik and Felix M.H. Villars, *Physics with Illustrative Examples from Medicine and Biology Volume 1 Mechanics*, Addison-Wesley, Reading, 1973, p. 5-116.

nuclear energy, or mass energy. Energy is always conserved in these processes although it may be converted from one form into another.

Any physical process can be characterized by an "initial state" that transforms into a "final state". Each form of energy E_i undergoes a change during this transformation,

$$\Delta E_i = E_{\text{final},i} - E_{\text{initial},i} \tag{C.1.1}$$

Conservation of energy means that the sum of these changes is zero,

$$\Delta E_1 + \Delta E_2 + \cdots = \sum_{i=1}^{N} \Delta E_i = 0 \tag{C.1.2}$$

Two critical points emerge. The first is that only change in energy has meaning. The initial or final energy is actually a meaningless concept. What we need to count is the change of energy and so we search for physical laws that determine how each form of energy changes. The second point is that we must account for all the ways energy can change. If we observe a process, and the changes in energy do not add up to zero, then the laws for energy transformations are either wrong or there is a new type of change of energy that we had not previously discovered. Some quantity is conserved in all processes and we call that *energy*. If we can quantify the changes of forms of energies then we have a very powerful tool to understand nature.

We will begin our analysis of conservation of energy by considering processes involving only a few forms of changing energy. We will make assumptions such as "ignore the effects of friction". This means that from the outset we assume that the change in heat energy is zero.

Energy is always conserved but sometimes we prefer to restrict our attention to a set of objects that we define to be our system. The rest of the universe acts as the surroundings. Our conservation of energy then becomes

$$\Delta E_{\text{system}} + \Delta E_{\text{surroundings}} = 0 \tag{C.1.3}$$

C.1.2 Kinetic Energy

Our first form of energy that we will study is the *kinetic energy* K, an energy associated with the motion of an object with mass m. Let's consider a car moving along a straight road (call this road the x-axis) with velocity $\vec{\mathbf{v}} = v_x \,\hat{\mathbf{i}}$. The speed v of the car is the magnitude of the velocity. The kinetic energy of the car is defined to be the positive scalar quantity

$$K = \frac{1}{2} m v^2 \tag{C.1.4}$$

Note that the kinetic energy is proportional to the square of the speed of the car. The SI unit for kinetic energy is $\left[\mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-2}\right]$; this combination units is defined to be a joule and is denoted by [J]. Thus $1\ \mathrm{J} \equiv 1\ \mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-2}$.

Let's consider a case in which our car changes velocity. For our initial state, the car moves with an initial velocity $\vec{\mathbf{v}}_0 = v_{x,0}\,\hat{\mathbf{i}}$ along the x-axis. For the final state (some time later), the car has changed its velocity and now moves with a final velocity $\vec{\mathbf{v}}_f = v_{x,f}\,\hat{\mathbf{i}}$. Therefore the change in the kinetic energy is

$$\Delta K = \frac{1}{2}m{v_f}^2 - \frac{1}{2}m{v_0}^2 \qquad \text{(C.1.5)}$$

C.2 Work and Power

C.2.1 Work Done by Constant Forces

We begin our discussion of the concept of work by analyzing the motion of a rigid body in one dimension acted on by constant forces. Let's consider an example of this type of motion: pushing a cup forward with a constant force along a desktop. When the cup changes velocity and hence kinetic energy, the sum of the forces acting on the cup must be non-zero according to Newton's Second Law. There are three forces involved in this motion, the applied pushing force $\vec{\mathbf{F}}_{\text{applied}}$, the contact force $\vec{\mathbf{C}} = \vec{\mathbf{N}} + \vec{\mathbf{f}}_{\mathrm{k}}$, and gravity, $\vec{\mathbf{F}}_{\text{grav}} = m\vec{\mathbf{g}}$. The force diagram is shown in Figure C.2.1.

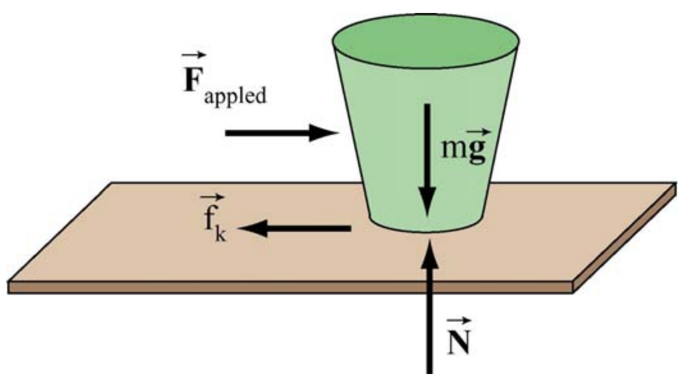

Figure C.2.1: Force diagram on a cup.

Let's choose our coordinate system so that the $+x$-direction is the direction of motion of the cup forward. Then the pushing force can be described by,

$$\vec{\mathbf{F}}_{\text{applied}} = F_{\text{applied},x}\,\hat{\mathbf{i}} \qquad \text{(C.2.1)}$$

Definition: Work done by a Constant Force

> *Suppose a body moves in a straight line from an initial point* x_0 *to a final point* x_f *so that the displacement of the cup is positive,* $\Delta x \equiv x_f - x_0 > 0$ *. The work W done by the constant force* $\vec{\mathbf{F}}_{\text{applied}}$ *acting on the body is the product of the component of the force* $F_{\text{applied},x}$ *and the displacement* Δx *,*

$$W_{\text{applied}} = F_{\text{applied},x}\,\Delta x \tag{C.2.2}$$

Work is a scalar quantity; it is not a vector quantity. The SI units for work are joules since $[1\ \text{N}\cdot\text{m}] = [1\ \text{J}]$. Note that work has the same dimension as kinetic energy.

If our applied force is along the direction of motion, both $F_{\text{applied},x} > 0$ and $\Delta x > 0$, so the work done is just the product of the magnitude of the applied force with the distance moved and is positive. We can extend the concept of work to forces that oppose the motion, such as friction.

In our example of the moving cup, the friction force is

$$\vec{\mathbf{f}}_{\text{k}} = f_x\,\hat{\mathbf{i}} = -\mu_{\text{k}} N\,\hat{\mathbf{i}} = -\mu_{\text{k}} mg\,\hat{\mathbf{i}} \tag{C.2.3}$$

Here the component of force is in the opposite direction as the displacement. The work done by the friction force is negative,

$$W_{\text{friction}} = -\mu_{\text{k}} mg\,\Delta x \tag{C.2.4}$$

Since the gravitational force is perpendicular to the motion of the cup, it has no component along the line of motion. Therefore, gravity does zero work on the cup when the cup is slid forward in the horizontal direction. The normal force is also perpendicular to the motion, hence does no work.

In summary, the gravitational force and the normal force do zero work, the pushing force does positive work, and the friction force does negative work.

C.2.2 Work and the Dot Product

A very important physical example of the dot product of two vectors is work. Recall that when a constant force acts on a mass that is moving along the x-axis, only the component of the force along that direction contributes to the work,

$$W = F_x\,\Delta x \tag{C.2.5}$$

For example, suppose we are pulling a mass along a horizontal surface with a force $\vec{\mathbf{F}}$. Choose coordinates such that horizontal direction is the x-axis and the force $\vec{\mathbf{F}}$ forms an angle β with the positive x-direction. In Figure C.2.2 we show the force vector $\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}}$ and the displacement vector $\Delta\vec{\mathbf{x}} = \Delta x\,\hat{\mathbf{i}}$. Note that Δx is the component of the displacement and hence can be greater, equal, or less than zero.

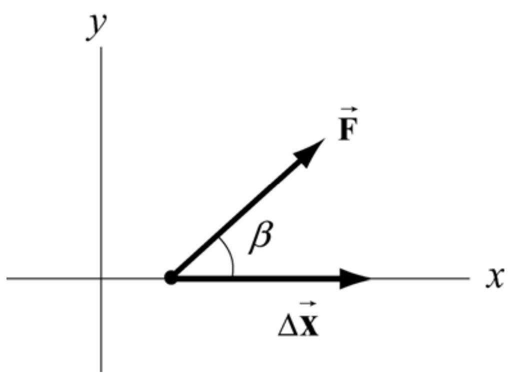

Figure C.2.2 Force and displacement vectors

Then the dot product between the force vector $\vec{\mathbf{F}}$ and the displacement vector $\Delta\vec{\mathbf{x}}$ is

$$\vec{\mathbf{F}} \cdot \Delta\vec{\mathbf{x}} = (F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}}) \cdot (\Delta x\,\hat{\mathbf{i}}) = F_x\,\Delta x \qquad \text{(C.2.7)}$$

This is the work done by the force,

$$\Delta W = \vec{\mathbf{F}} \cdot \Delta\vec{\mathbf{x}} \qquad \text{(C.2.9)}$$

The force $\vec{\mathbf{F}}$ forms an angle β with the positive x-direction. The angle β takes values within the range $-\pi \le \beta \le \pi$. Since the x-component of the force is $F_x = F\cos\beta$ where $F = |\vec{\mathbf{F}}|$ denotes the magnitude of $\vec{\mathbf{F}}$, the work done by the force is

$$W = \vec{\mathbf{F}} \cdot \Delta\vec{x} = (F\cos\beta)\Delta x \qquad \text{(C.2.10)}$$

C.2.3 Work done by Non-Constant Forces

Consider a mass moving in the x-direction under the influence of a non-uniform force that is pointing in the x-direction, $\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}}$. The mass moves from an initial position x_0 to a final position x_f .In order to calculate the work done by a non-uniform force, we will divide up the displacement into a large number N of small displacements Δx_i where the index i denotes the ith displacement and takes on integer values from 1 to N, with

$x_N = x_f$. Let $(F_x)_i$ denote the average value of the x-component of the force in the interval $\left[x_{i-1}, x_i\right]$. For the i th displacement, the contribution to the work is

$$\Delta W_i = (F_x)_i \, \Delta x_i \tag{C.2.11}$$

This contribution is a scalar so we add up these scalar quantities to get the total work;

$$W_N = \sum_{i=1}^{i=N} \Delta W_i = \sum_{i=1}^{i=N} (F_x)_i \, \Delta x_i \tag{C.2.13}$$

This depends on the number of divisions N . In order to define a quantity that is independent of the divisions, take the limit as $N \to \infty$ and $\left|\Delta x_i\right| \to 0$. Then the work is

$$W = \lim_{\substack{N\to\infty \\ \Delta x_i \to 0}} \sum_{i=1}^{i=N} (F_x)_i \, \Delta x_i = \int_{x=x_0}^{x=x_f} F_x \, dx \tag{C.2.15}$$

This last expression is the definition of the integral of the x-component of the force with respect to the parameter x. In Figure C.2.3, the graph of the x-component of the force with respect to the parameter x is shown. The work integral is the area under this curve.

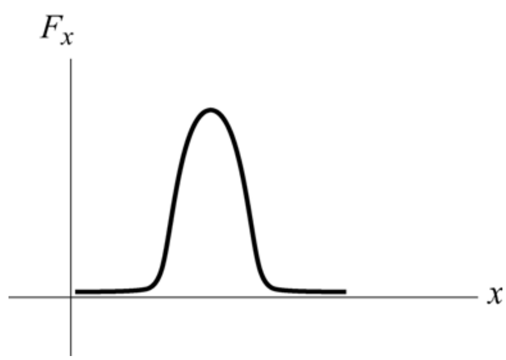

Figure C.2.3 Graph of x-component of the force as a function of x

C.2.4 Work Done Along an Arbitrary Path

Now suppose that a non-constant force $\vec{\mathbf{F}}$ acts on an object of mass m while the object is moving on a three-dimensional curved path. The position vector of the particle at time t with respect to a choice of origin is $\vec{\mathbf{r}}(t)$. In Figure C.2.4, the orbit of the object is shown of the object for a time interval $\left[t_0, t_f\right]$, moving from an initial position $\vec{\mathbf{r}}_0 \equiv \vec{\mathbf{r}}(t = t_0)$ at time $t = t_0$ to a final position $\vec{\mathbf{r}}_f \equiv \vec{\mathbf{r}}(t = t_f)$ at time $t = t_f$.

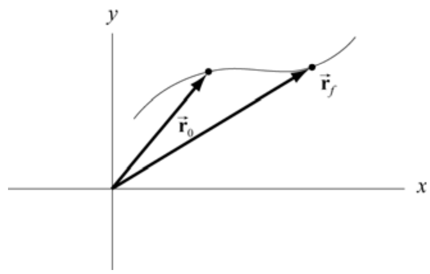

Figure C.2.4 Orbit of the mass.

We divide the time interval $\left[t_0, t_f\right]$ into N small pieces with $t_N = t_f$. Each individual piece is labeled by the index i taking on integer values from 1 to N. Consider two position vectors $\vec{\mathbf{r}}_i \equiv \vec{\mathbf{r}}(t = t_i)$ and $\vec{\mathbf{r}}_{i-1} \equiv \vec{\mathbf{r}}(t = t_{i-1})$ marking the i and $i-1$ position. The displacement $\Delta\vec{\mathbf{r}}_i$ is then $\Delta\vec{\mathbf{r}}_i = \vec{\mathbf{r}}_i - \vec{\mathbf{r}}_{i-1}$.

Let $\vec{\mathbf{F}}_i$ denote the average force acting on the mass during the interval $\left[t_{i-1}, t_i\right]$. We can locate the force in the middle of the path between $\vec{\mathbf{r}}_i$ and $\vec{\mathbf{r}}_{i-1}$. The average work ΔW_i done by the force during the time interval $\left[t_{i-1}, t_i\right]$ is the dot product between the average force vector and the displacement vector corresponding to the product of the component of the average force in the direction of the displacement with the displacement,

$$\Delta W_i = \vec{\mathbf{F}}_i \cdot \Delta\vec{\mathbf{r}}_i \tag{C.2.16}$$

The force and the displacement vectors for the time interval $\left[t_{i-1}, t_i\right]$ are shown in Figure C.2.5.

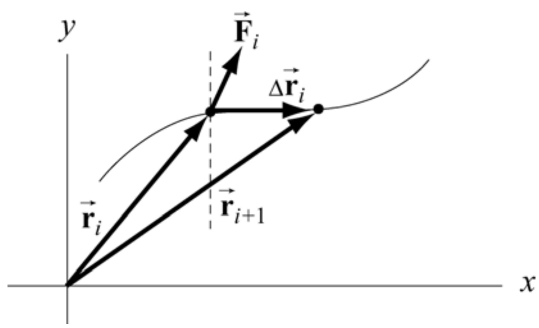

Figure C.2.5 Infinitesimal work diagram

The work done is found by adding these scalar contributions to the work for each interval $\left[t_{i-1}, t_i\right]$, for $i = 1$ to N,

$$W_N = \sum_{i=1}^{i=N} \Delta W_i = \sum_{i=1}^{i=N} \vec{\mathbf{F}}_i \cdot \Delta\vec{\mathbf{r}}_i \tag{C.2.17}$$

We would like to define work in a manner that is independent of the way we divide the interval so we take the limit as $N \to \infty$ and $|\Delta \vec{\mathbf{r}}_i| \to 0$. In this limit as the intervals become smaller and smaller, the distinction between the average force and the actual force becomes vanishingly small. Thus if this limit exists and is well defined, then the work done by the force is

$$W = \lim_{\substack{N \to \infty \\ |\Delta \vec{\mathbf{r}}_i| \to 0}} \sum_{i=1}^{i=N} \vec{\mathbf{F}}_i \cdot \Delta \vec{\mathbf{r}}_i = \int_{r_0}^{r_f} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \qquad \text{(C.2.19)}$$

Notice that this summation involves adding scalar quantities. This limit is called the "*line integral of the tangential component*" of the force $\vec{\mathbf{F}}$. The symbol $d\vec{\mathbf{r}}$ is called the "*infinitesimal vector line element*". At time t, $d\vec{\mathbf{r}}$ is tangent to the orbit of the mass and is the limit of the displacement vector $\Delta \vec{\mathbf{r}} = \vec{\mathbf{r}}(t + \Delta t) - \vec{\mathbf{r}}(t)$ as Δt approaches zero.

In general, this line integral depends on the particular path the object takes between the initial position $\vec{\mathbf{r}}_0$ and the final position $\vec{\mathbf{r}}_f$. The reason is that the force $\vec{\mathbf{F}}$ is non-constant in space and the contribution to the work can vary over different paths in space.

We can represent this integral explicitly in a coordinate system by specifying the infinitesimal vector line element $d\vec{\mathbf{r}}$ and then explicitly computing the dot product. For example in Cartesian coordinates the line element is

$$d\vec{\mathbf{r}} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}} \qquad \text{(C.2.20)}$$

where dx, dy, and dz represent arbitrary displacements in the x, y, and z -directions respectively as seen in Figure C.2.6.

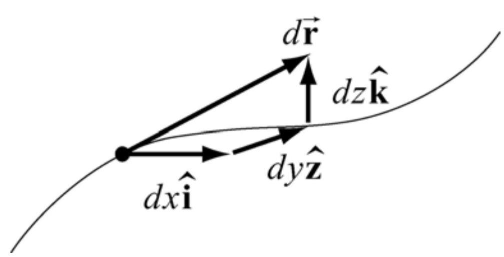

Figure C.2.6 Line element in Cartesian coordinates

The force vector can be represented in vector notation by

$$\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}} + F_z\,\hat{\mathbf{k}} \qquad \text{(C.2.21)}$$

Then the infinitesimal work is the dot product

$$dW = \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = (F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}} + F_z\hat{\mathbf{k}}) \cdot (dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}) \tag{C.2.23}$$

$$dW = F_x\,dx + F_y\,dy + F_z\,dz \tag{C.2.24}$$

so the total work is

$$W = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} \left(F_x\,dx + F_y\,dy + F_z\,dz\right) = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} F_x\,dx + \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} F_y\,dy + \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} F_z\,dz \tag{C.2.25}$$

The above equation shows that W consists of three separate integrals. In order to calculate these integrals in general we need to know the specific path the object takes.

C.2.5 Power

Definition: Power by a Constant Force:

Suppose that an applied force $\vec{\mathbf{F}}_{\text{applied}}$ acts on a body during a time interval Δt, and displaces the body in the x-direction by an amount Δx. The work done, ΔW, during this interval is

$$\Delta W = F_{\text{applied},x}\,\Delta x \tag{C.2.26}$$

where $F_{\text{applied},x}$ is the x-component of the applied force.

The average power of the applied force is defined to be the rate of doing work

$$P_{\text{ave}} = \frac{\Delta W}{\Delta t} = \frac{F_{\text{applied},x}\,\Delta x}{\Delta t} = F_{\text{applied},x}\,v_{x,\text{ave}} \tag{C.2.27}$$

.

The average power delivered to the body is equal to the component of the force in the direction of motion times the average velocity of the body.

Power is a scalar quantity and can be positive, zero, or negative depending on the sign of work. The SI units of power are called watts $[\text{W}]$ and $[1\text{W}] \equiv [1\text{J} \cdot \text{s}^{-1}] \equiv [1\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-3}]$.

The *instantaneous power* at time t is defined to be the limit of the average power as the time interval $[t, \Delta t]$ approaches zero,

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \lim_{\Delta t \to 0} \frac{F_{\text{applied},x}\,\Delta x}{\Delta t} = F_{\text{applied},x} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = F_{\text{applied},x}\,v_x \tag{C.2.28}$$

the instantaneous power of a constant applied force is the product of the force and the instantaneous velocity of the moving object.

C.3 Work and Energy

C.3.1 Work-Kinetic Energy Theorem

There is a connection between the total work done on an object and the change of kinetic energy. Non-zero total work implies that the total force acting on the object is non-zero. Therefore the object will accelerate. When the total work done on an object is positive the object will increase its speed. When the work done is negative, the object will decrease its speed. When the total work done is zero, the object will maintain a constant speed. In fact we have a more precise result, the total work done by all the applied forces on an object is equal to the change in kinetic energy of the object.

$$W_{\text{total}} = \Delta K = \frac{1}{2} m v_f^{\,2} - \frac{1}{2} m v_0^{\,2} \tag{C.3.1}$$

C.3.2 Work-Kinetic Energy Theorem for Non-Constant Forces

The work-kinetic energy theorem holds as well for a non-constant force. Recall that the definition of work done by a non-constant force in moving an object along the x-axis from an initial position x_0 to the final position x_f is given by

$$W = \int_{x_0}^{x_f} F_x \, dx \tag{C.3.2}$$

where F_x is the component of the force in the x-direction. According to Newton's Second Law,

$$F_x = m \frac{dv_x}{dt} \tag{C.3.3}$$

Therefore the work integral can be written as

$$W = \int_{x_0}^{x_f} F_x \, dx = \int_{x_0}^{x_f} m \frac{dv_x}{dt} dx = \int_{x_0}^{x_f} m \frac{dx}{dt} dv_x \tag{C.3.4}$$

Since the x-component of the velocity is defined as $v_x = dx/dt$, the work integral becomes

$$W = \int_{v_{x,0}}^{v_{x,f}} m \frac{dx}{dt} dv_x = \int_{v_{x,0}}^{v_{x,f}} m v_x \, dv_x \tag{C.3.5}$$

Note that the limits of the integral have now be changed. Instead of integrating from the initial position x_0 to the final position x_f, the limits of integration are from the initial x-component of the velocity $v_{x,0}$ to the final x-component of the velocity $v_{x,f}$. Since

$$d\left(\frac{1}{2}mv_x^2\right) = mv_x\, dv_x \tag{C.3.6}$$

the integral is

$$W = \int_{v_{x,0}}^{v_{x,f}} mv_x\, dv_x = \int_{v_{x,0}}^{v_{x,f}} d\left(\frac{1}{2}mv_x^2\right) \tag{C.3.7}$$

It follows that

$$W = \int_{v_{x,0}}^{v_{x,f}} mv_x\, dv_x = \int_{v_{x,0}}^{v_{x,f}} d\left(\frac{1}{2}mv_x^2\right) = \frac{1}{2}mv_{x,f}^2 - \frac{1}{2}mv_{x,0}^2 = \Delta K \tag{C.3.8}$$

C.3.3 Work-Kinetic Energy Theorem for a Non-Constant Force in Three Dimensions

The work energy theorem generalized to three-dimensional motion. Suppose under the action of an applied force, an object changes its velocity from an initial velocity

$$\vec{\mathbf{v}}_0 = v_{x,0}\,\hat{\mathbf{i}} + v_{y,0}\,\hat{\mathbf{j}} + v_{z,0}\,\hat{\mathbf{k}} \tag{C.3.9}$$

to a final velocity

$$\vec{\mathbf{v}}_f = v_{x,f}\,\hat{\mathbf{i}} + v_{y,f}\,\hat{\mathbf{j}} + v_{z,f}\,\hat{\mathbf{k}} \tag{C.3.10}$$

The kinetic energy is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(v_x^2 + v_y^2 + v_z^2\right) \tag{C.3.11}$$

Therefore the change in kinetic energy is

$$\Delta K = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2 = \frac{1}{2}m\left({v_{x,f}}^2 + {v_{y,f}}^2 + {v_{z,f}}^2\right) - \frac{1}{2}m\left({v_{x,0}}^2 + v_{y,0}^2 + v_{z,0}^2\right) \tag{C.3.12}$$

The work done by the force in three dimensions is

$$W = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}_f} F_x \, dx + \int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}_f} F_y \, dy + \int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}_f} F_z \, dz \qquad \text{(C.3.13)}$$

As before ,we can apply Newton's Second Law to each integral separately using

$$F_x = m\frac{dv_x}{dt}, \quad F_y = m\frac{dv_y}{dt}, \quad F_z = m\frac{dv_z}{dt} \qquad \text{(C.3.14)}$$

The work is then

$$\begin{aligned} W &= \int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}_f} m\frac{dv_x}{dt} dx + \int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}_f} m\frac{dv_y}{dt} dy + \int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}_f} m\frac{dv_z}{dt} dz \\ &= \int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}_f} mdv_x \frac{dx}{dt} + \int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}_f} mdv_y \frac{dy}{dt} + \int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}_f} mdv_z \frac{dz}{dt} \end{aligned} \qquad \text{(C.3.15)}$$

This becomes

$$W = \int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}_f} m\,dv_x v_x + \int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}_f} m\,dv_y v_y + \int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}_f} m\,dv_z v_z \qquad \text{(C.3.16)}$$

These integrals can be integrated explicitly yielding the work-kinetic energy theorem

$$W = \left(\frac{1}{2}mv_{x,f}^2 - \frac{1}{2}mv_{x,0}^2\right) + \left(\frac{1}{2}mv_{y,f}{}^2 - \frac{1}{2}mv_{y,0}{}^2\right) + \left(\frac{1}{2}mv_{z,f}^2 - \frac{1}{2}mv_{z,0}^2\right) = \Delta K \quad \text{(C.3.17)}$$

C.3.4 Time Rate of Change of Kinetic Energy

In one dimension, the time rate of change of the kinetic energy,

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv_x^2\right) = mv_x\frac{dv_x}{dt} = mv_x a_x = F_x v_x \qquad \text{(C.3.18)}$$

since by Newton's Second Law,

$$F_x^{\text{total}} = ma_x \qquad \text{(C.3.19)}$$

the time derivative of the kinetic energy is equal to the instantaneous power delivered to the body,

$$\frac{dK}{dt} = F_x v_x = P \qquad \text{(C.3.20)}$$

The generalization to three dimensions becomes

$$\frac{dK}{dt} = F_x v_x + F_y v_y + F_z v_z = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}} = P \tag{C.3.21}$$