

**MASSACHUSETTS INSTITUTE OF TECHNOLOGY**  
**Department of Physics**

**8.02**

**Review D: Potential Energy and the Conservation of Mechanical Energy**

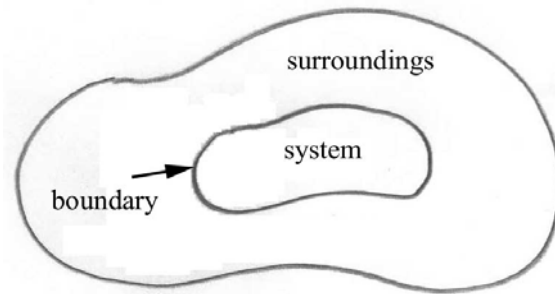
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# Potential Energy and the Conservation of Mechanical Energy

## D.1 Conservative and Non-conservative Force

### D.1.1 Introduction

A “physical system” consists of a well-defined set of bodies that are interacting by means of forces. Any bodies that lie outside the boundary of the system reside in the “surroundings”. A state of the system is a set of measurable physical quantities that completely characterize the system. Figure 1 shows this division into system, boundary, and surroundings.



**Figure D.1.1:** system, boundary, and surroundings.

Up to now we have analyzed the dynamical evolution in time of our system under the action of forces using Newton’s Laws of Motion. We shall now introduce the concept of **Conservation of Energy** in order to analyze the change of state of a system.

*Definition: Change of Energy*

*The total change in energy of a system and its surroundings between the final state and the initial state is zero,*

$$\Delta E_{\text{total}} = \Delta E_{\text{system}} + \Delta E_{\text{surroundings}} = 0 \quad (\text{D.1.1})$$

Our quest is then to identify experimentally every type of change of energy for all physical processes and verify that energy is conserved. Can we really play this “zero sum” game? Is there any physical content to this concept of change of energy? The answer is that experimentally we can identify all the changes in energy. One important point to keep in mind is that if we add up all the changes in energy and do not arrive at a zero sum then we have an open scientific problem: find the missing change in energy!

Our first example of this type of “energy accounting” involves *mechanical energy*. There will be of two types of mechanical energy, *kinetic energy* and *potential energy*. Our first task is to define what we mean by the change of the potential energy of a system.

## D.1.2 Conservative and Non-Conservative Forces

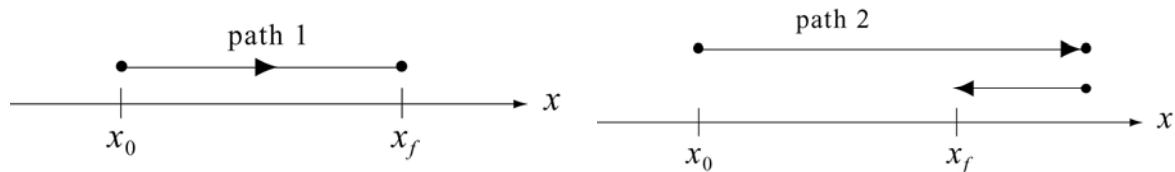
We have defined the work done by a force  $\vec{F}$  on an object which moves from an initial point  $x_0$  to a final point  $x_f$  (with displacement  $\Delta x \equiv x_f - x_0$ ), as the product of the component of the force  $F_x$  with the displacement  $\Delta x$ ,

$$W = F_x \Delta x \quad (\text{D.1.2})$$

Does the work done on the object by the force depend on the path taken by the object? For a force such as friction we expect that the work should depend on the path. Let's compare two paths from an initial point  $x_0$  to a final point  $x_f$ . The first path is a straight-line path with no reversal of direction. The second path goes past  $x_f$  some distance and then comes back to  $x_f$  (Figure 2). Since the force of friction always opposes the motion, the work done by friction is negative,

$$W_{\text{friction}} = -\mu_k N \Delta x < 0 \quad (\text{D.1.3})$$

Therefore the total work depends on the total distance traveled, not the displacement. Hence, the magnitude of the work done along the second path is greater than the magnitude of the work done along the first path.

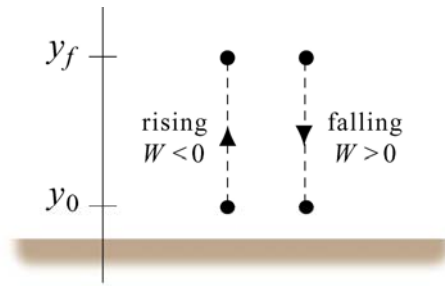


**Figure D.1.2** Two different paths from  $x_0$  to  $x_f$

Let's consider the motion of an object under the influence of a gravitational force near the surface of the earth. The gravitational force always points downward, so the work done by gravity only depends on the change in the vertical position,

$$W_{\text{grav}} = F_{\text{grav},y} \Delta y = -mg \Delta y \quad (\text{D.1.4})$$

Therefore, when an object falls, the work done by gravity is positive and when an object rises the work done by gravity is negative. Suppose an object first rises and then falls, returning to the original starting height. The positive work done on the falling portion exactly cancels the negative work done on the rising portion, (Figure 3). The total work done is zero! Thus, the gravitational work done between two points will not depend on the path taken, but only on the initial and final positions.



**Figure D.1.3** Gravitational work cancels.

These two examples typify two fundamentally different types of forces and their contribution to work. In the first example of sliding friction, the work done depends on the path.

***Definition: Non-conservative Force***

*Whenever the work done by a force in moving an object from an initial point to a final point depends on the path, then the force is called a non-conservative force.*

In the second example of free fall, the work done by the gravitational force is independent of the path.

***Definition: Conservative Force***

*Whenever the work done by a force in moving an object from an initial point to a final point is independent of the path, then the force is called a conservative force.*

**D.2 Potential Energy**

**D.2.1 Change in Potential Energy**

Let's consider an example in which only conservative forces are acting on a system, for example the gravitational force considered above. The key concept for a conservative force such as gravitation is that the work done by the gravitational force on an object only depends on the change of height of the object. Instead of analyzing the forces acting on an object, we introduce the idea of changes in the kinetic and potential energy of a physical system. This allows us use introduce conservation of energy to describe motion.

In the case of an object falling near the surface of the earth, our system consists of the earth and the object and the state is described by the height of the object above the surface of the earth. The initial state is at a height  $y_0$  above the surface of the earth and the final state is at a height  $y_f$  above the surface of the earth.

The change in kinetic energy between the initial and final states for both the earth and the object is

$$\Delta K_{\text{system}} = \Delta K_{\text{earth}} + \Delta K_{\text{object}} = \left( \frac{1}{2} m v_{\text{earth},f}^2 - \frac{1}{2} m v_{\text{earth},0}^2 \right) + \left( \frac{1}{2} m v_{\text{object},f}^2 - \frac{1}{2} m v_{\text{object},0}^2 \right) \quad (\text{D.2.1})$$

The change of kinetic energy of the earth due to the gravitational interaction between the earth and the object is negligible. So the total change in kinetic energy of the system is approximately equal to the change in kinetic energy of the object,

$$\Delta K_{\text{system}} \cong \Delta K_{\text{object}} = \left( \frac{1}{2} m v_{\text{object},f}^2 - \frac{1}{2} m v_{\text{object},0}^2 \right) \quad (\text{D.2.2})$$

We will enlarge our idea of change of energy by defining a new type of change of energy, a change of gravitational potential energy  $\Delta U_{\text{system}}$ , so that the total change of both kinetic and potential energy is zero,

$$\Delta K_{\text{system}} + \Delta U_{\text{system}} = 0 \quad (\text{D.2.3})$$

In what follows, we'll look at arbitrary conservative forces to obtain a general result, and then return to the example of gravity, along with other perhaps familiar examples.

### D.2.2 Definition: Change in Potential Energy

*The change in potential energy of a body associated with a conservative force is the negative of the work done by the conservative force in moving the body along any path connecting the initial position to the final position.*

$$\Delta U_{\text{system}} = -W_{\text{cons}} = - \int_A^B \vec{\mathbf{F}}_{\text{cons}} \cdot d\vec{\mathbf{r}} \quad (\text{D.2.4})$$

where  $A$  and  $B$  are used to represent the initial and final positions of the body.

This definition only holds for conservative forces because the work done by a conservative force does not depend on the path and only depends on the initial and final positions.

The work-kinetic energy theorem states that if the work done by the conservative force is the *only* work done, and hence is the total work, then that work is equal to the change in kinetic energy of the system;

$$\Delta K_{\text{system}} = W^{\text{total}} = W_{\text{cons}} \quad (\text{D.2.5})$$

The total change in mechanical energy is defined to be the sum of the changes of the kinetic and the potential energies,

$$\Delta E_{\text{mechanical}} = \Delta K_{\text{system}} + \Delta U_{\text{system}} \quad (\text{D.2.6})$$

From both the definition of change of potential energy,  $\Delta U_{\text{system}} = -W_{\text{cons}}$ , and the work-kinetic energy theorem, the total change in the mechanical energy is zero,

$$\Delta E_{\text{mechanical}} = W_{\text{cons}} + (-W_{\text{cons}}) = 0 \quad (\text{D.2.7})$$

This expression is known as the **conservation of mechanical energy**.

### D.2.3 Several Conservative Forces

When there are several conservative forces acting on the system we define a separate change in potential energy for the work done by each conservative force,

$$\Delta U_{\text{system},i} = -W_i = -\int_A^B \vec{\mathbf{F}}_{\text{cons},i} \cdot d\vec{\mathbf{r}} \quad (\text{D.2.8})$$

The total work done is just the sum of the individual work,

$$W_{\text{cons}}^{\text{total}} = W_{\text{cons},1} + W_{\text{cons},2} + \dots \quad (\text{D.2.9})$$

The sum of the change in potential for the system is

$$\Delta U_{\text{system}}^{\text{total}} = \Delta U_{\text{system},1} + \Delta U_{\text{system},2} + \dots \quad (\text{D.2.10})$$

Therefore, the total change in potential energy of the system is equal to the negative of the total work done

$$\Delta U_{\text{system}}^{\text{total}} = -W_{\text{cons}}^{\text{total}} = -\sum_i \int_A^B \vec{\mathbf{F}}_{\text{cons},i} \cdot d\vec{\mathbf{r}} \quad (\text{D.2.11})$$

### D.2.4 Change in Mechanical Energy and Conservation of Mechanical Energy

We now define the mechanical energy function for a system,

$$E_{\text{mechanical}} = K + U \quad (\text{D.2.12})$$

where  $K$  is the kinetic energy and  $U$  is the potential energy.

Then the change in mechanical energy when the system goes from an initial state to a final state is

$$\Delta E_{\text{mechanical}} \equiv E_{\text{mechanical},f} - E_{\text{mechanical},0} = (K_f + U_f) - (K_0 + U_0) \quad (\text{D.2.13})$$

When there are no non-conservative forces acting on the system, the total mechanical energy of the system is conserved, and

$$E_{\text{mechanical},f} = E_{\text{mechanical},0} \quad (\text{D.2.14})$$

or equivalently

$$(K_f + U_f) = (K_0 + U_0) \quad (\text{D.2.15})$$

### D.2.5 Non-Conservation of Mechanical Energy

The total force acting on a system may, in general, consist of a vector sum of conservative forces and non-conservative forces,

$$\vec{\mathbf{F}}^{\text{total}} = \sum_i \vec{\mathbf{F}}_{\text{cons},i} + \sum_j \vec{\mathbf{F}}_{\text{non-cons},j} \quad (\text{D.2.16})$$

The work done by the total non-conservative force,

$$\vec{\mathbf{F}}_{\text{non-cons}}^{\text{total}} = \sum_j \vec{\mathbf{F}}_{\text{non-cons},j} \quad (\text{D.2.17})$$

between two points  $A$  and  $B$  is a path-dependent quantity which we denote by

$$W_{\text{non-cons}} [A, B] = \int_A^B \vec{\mathbf{F}}_{\text{non-cons}}^{\text{total}} \cdot d\vec{\mathbf{r}} \quad (\text{D.2.18})$$

Then the work done by the total force is

$$\begin{aligned} W^{\text{total}} [A, B] &= \int_A^B \vec{\mathbf{F}}^{\text{total}} \cdot d\vec{\mathbf{r}} = \int_A^B \left( \sum_i \vec{\mathbf{F}}_{\text{cons},i} + \sum_j \vec{\mathbf{F}}_{\text{non-cons},j}^{\text{total}} \right) \cdot d\vec{\mathbf{r}} \\ &= \sum_i \int_A^B \vec{\mathbf{F}}_{\text{cons},i} \cdot d\vec{\mathbf{r}} + W_{\text{non-cons}} [A, B] \\ &= -\Delta U^{\text{total}} + W_{\text{non-cons}} [A, B] \end{aligned} \quad (\text{D.2.19})$$

The work-kinetic energy theorem states that the work done by the total force is equal to the change in kinetic energy,

$$W_{\text{total}} = \Delta K = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2 \quad (\text{D.2.20})$$

Thus

$$\Delta K = -\Delta U^{\text{total}} + W_{\text{non-cons}} [A, B] \quad (\text{D.2.21})$$

We define the change in mechanical energy of the system, between the states represented by  $A$  and  $B$ ,

$$\Delta E_{\text{mechanical}}^{\text{total}} [A, B] \equiv \Delta K + \Delta U^{\text{total}} \quad (\text{D.2.22})$$

as the sum of the change of the kinetic energy and change of the total potential energy. This change  $\Delta E_{\text{mechanical}}^{\text{total}} [A, B]$  is independent of the path between the states  $A$  and  $B$ .

Therefore the total change in the mechanical energy of the system between the states  $A$  and  $B$  is equal to the work done by the non-conservative forces along the chosen path connecting the points  $A$  and  $B$ ,

$$\Delta E_{\text{mechanical}}^{\text{total}} = W_{\text{non-cons}} [A, B] \quad (\text{D.2.23})$$

When there are no non-conservative forces acting on the system, then the total mechanical energy of the system is conserved,

$$\Delta E_{\text{mechanical}}^{\text{total}} = 0 \quad (\text{D.2.24})$$

### D.3 Worked Examples: Calculation of the Change in Potential Energy

There are four examples of conservative forces for which we will calculate the work done by the force; constant gravity near the surface of the earth, the ideal spring force, the universal gravitational force between two point-like objects, and the electric force between charges (this last calculation will not be presented in this review).

For each of these forces we can describe and calculate the change in potential energy according to our definition

$$\Delta U_{\text{system}} = -W_{\text{cons}} = -\int_A^B \vec{\mathbf{F}}_{\text{cons}} \cdot d\vec{\mathbf{r}} \quad (\text{D.3.1})$$

In addition, we would like to choose a “zero point” for the potential energy. This is a point where we define potential energy to be zero and all changes in potential energy are made relative to this point. This point will in general be different for different forces.

#### D.3.1 Example 1: Change in Gravitational Potential Energy – Uniform Gravity

Let’s choose a coordinate system with the origin at the surface of the earth and the  $+y$ -direction pointing away from the center of the earth. Suppose an object of mass  $m$  moves



from an initial point  $y_0$  to a final point  $y_f$ . When an object is placed in the gravitational field near the surface of the earth, the gravitational force on the object is given by

$$\vec{\mathbf{F}}_{\text{grav}} = m\vec{\mathbf{g}} = F_{\text{grav},y} \hat{\mathbf{j}} = -mg \hat{\mathbf{j}} \quad (\text{D.3.2})$$

The work done by the gravitational force on the object is then

$$W_{\text{grav}} = F_{\text{grav},y} \Delta y = -mg \Delta y \quad (\text{D.3.3})$$

and the change in potential energy is given by

$$\Delta U_{\text{system}} = -W_{\text{grav}} = mg \Delta y = mg y_f - mg y_0 \quad (\text{D.3.4})$$

We will introduce a potential energy function  $U$  given by

$$\Delta U_{\text{system}} \equiv U_f - U_0.$$

Suppose the mass starts out at the surface of the earth with  $y_0 = 0$ . Then the change in potential energy is

$$\Delta U_{\text{system}} \equiv U(y_f) - U(y_0 = 0) = mg y_f - mg y_0 \quad (\text{D.3.5})$$

We are free to choose the zero point for the potential energy anywhere we like since change in potential energy only depends on the displacement,  $\Delta y$ . We have some flexibility to adapt our choice of zero for the potential energy to best fit a particular problem. In the above expression for the change of potential energy, let  $y_f \equiv y$  be an arbitrary point and  $y_0 = 0$  be the origin. Then we can choose the zero reference point for the potential energy to be at the origin,

$$U(y) - 0 = mg y - 0 \text{ or } U(y) = mgy \text{ with } U(y_0 = 0) = 0 \quad (\text{D.3.6})$$

### D.3.2 Example 2: Hooke's Law Spring-mass system

Consider a spring-object system lying on a frictionless horizontal surface with one end of the spring fixed to a wall and the other end attached to a object of mass  $m$ . Choose the origin at the position of the center of the object when the spring is unstretched, the equilibrium position. Let  $x$  be the displacement of the object from the origin. We choose the  $+\hat{\mathbf{i}}$  unit vector to point in the direction the object moves when the spring is being stretched. Then the spring force on a mass is given by

$$\vec{\mathbf{F}} = F_x \hat{\mathbf{i}} = -kx \hat{\mathbf{i}} \quad (\text{D.3.7})$$

The work done by the spring force on the mass is

$$W = \int_{x=x_0}^{x=x_f} (-kx) dx = -\frac{1}{2}k(x_f^2 - x_0^2) \quad (\text{D.3.8})$$

Therefore the change in potential energy in the spring-object system in moving the object from an initial position  $x_0$  from equilibrium to a final position  $x_f$  from equilibrium is

$$\Delta U_{\text{spring}} \equiv U_{\text{spring}}(x_f) - U_{\text{spring}}(x_0) = -W_{\text{spring}} = \frac{1}{2}k(x_f^2 - x_0^2) \quad (\text{D.3.9})$$

For the spring- object system , there is an obvious choice of position where the potential energy is zero, the equilibrium position of the spring- object,

$$U_{\text{spring}}(x = 0) \equiv 0 \quad (\text{D.3.10})$$

Then with this choice of zero point  $x_0 = 0$  , with  $x_f = x$  , an arbitrary extension or compression of a spring- object system, the potential energy function is given by

$$U_{\text{spring}}(x) = \frac{1}{2}k x^2 \quad (\text{D.3.11})$$

### D.3.3 Example 3: Inverse Square Gravitational Force

Consider two objects of masses  $m_1$  and  $m_2$  that are separated by a center-to-center distance  $r$  . The gravitational force between the two objects is given by

$$\vec{\mathbf{F}}_{m_1, m_2} = -\frac{Gm_1 m_2}{r^2} \hat{\mathbf{r}} \quad (\text{D.3.12})$$

where  $\hat{\mathbf{r}}$  is the unit vector directed along the line joining the masses. This expression assumes that the objects may be approximated as spherical.

The work done by this gravitational force in moving the two objects from an initial position in which the center of mass of the two objects are a distance  $r_0$  apart to a final position in which the center of mass of the two objects are a distance  $r_f$  apart is given by

$$W = \int_{r_0}^{r_f} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{r_0}^{r_f} \left( -\frac{Gm_1 m_2}{r^2} \right) dr \quad (\text{D.3.13})$$

Upon evaluation of this integral, we have for the work

$$W = \int_{r_0}^{r_f} \left( -\frac{Gm_1m_2}{r^2} \right) dr = \frac{Gm_1m_2}{r} \Big|_{r_0}^{r_f} = Gm_1m_2 \left( \frac{1}{r_f} - \frac{1}{r_0} \right) \quad (\text{D.3.14})$$

Therefore the change in potential energy of the system is

$$\Delta U_{\text{grav}} \equiv U_{\text{grav}}(r_f) - U_{\text{grav}}(r_0) = -W_{\text{gravity}} = -Gm_1m_2 \left( \frac{1}{r_f} - \frac{1}{r_0} \right) \quad (\text{D.3.15})$$

We now choose our reference point for the zero of the potential energy to be at infinity,  $r_0 = \infty$ ,

$$U_{\text{grav}}(r_0 = \infty) \equiv 0 \quad (\text{D.3.16})$$

The reason for this choice is that the term  $1/r_0$  in the expression for the change in potential energy vanishes when  $r_0 = \infty$ . Then the gravitational potential energy function for the two objects when their center of mass to center of mass distance is  $r_f = r$ , becomes

$$U_{\text{grav}}(\mathbf{r}) = -\frac{Gm_1m_2}{r},$$

#### D.4 Energy Diagrams: One-Dimensional Example: Spring Forces

From the above derivation of the potential energy of a compressed or extended ideal spring of spring constant  $k$ , we have

$$U_{\text{spring}}(x) - U_{\text{spring}}(x_0) = -W_{\text{spring}} = \frac{1}{2}k(x^2 - x_0^2) \quad (\text{D.4.1})$$

Taking  $x_0 = 0$  and setting  $U_{\text{spring}}(x_0) = 0$ , and dropping the subscript “spring”, we have the simplified

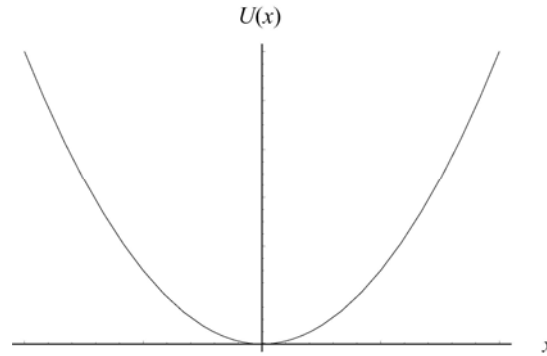
$$U(x) = \frac{1}{2}kx^2 \quad (\text{D.4.2})$$

It follows immediately that

$$\frac{d}{dx}U(x) = kx = -F_x \quad (\text{D.4.3})$$

and indeed, this expression follows immediately from the definition of potential energy in terms of the force in one dimension, regardless of the functional form of  $F_x$

In Figure D.4.1 we plot the potential energy function for the spring force as function of  $x$  with  $U(x_0 = 0) \equiv 0$ .



**Figure D.4.1:** graph of potential energy function for the spring

The minimum of the potential energy function occurs at the point where the first derivative vanishes

$$\frac{dU(x)}{dx} = 0 \quad (\text{D.4.4})$$

For the spring force

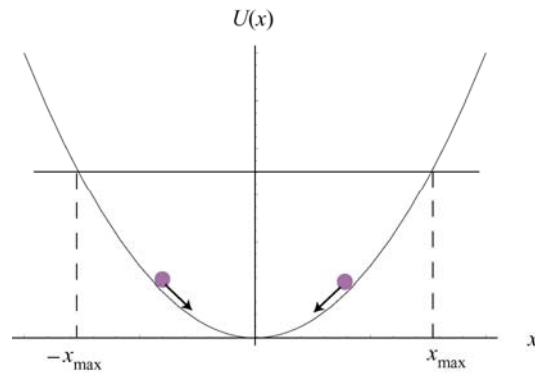
$$0 = \frac{dU(x)}{dx} = kx \quad (\text{D.4.5})$$

implies that the minimum occurs at  $x = 0$  (the second derivative  $d^2U/dx^2 = k > 0$  shows that this point is a minimum).

Since the force is the negative derivative of the potential energy, and this derivative necessarily vanishes at the minimum of the potential, we have that the spring force is zero at the minimum  $x = 0$  agreeing with our force law,  $F_x|_{x=0} = -kx|_{x=0} = 0$ .

In general, if the potential energy function has a minimum at some point then the force is zero at that point. If the object is extended a small distance  $x > 0$  away from equilibrium, the slope of the potential energy function is positive,  $dU(x)/dx > 0$  ( $d^2U/dx^2 > 0$  for a minimum); hence the component of the force is negative since  $F_x = -dU(x)/dx < 0$ . Thus, the object experiences a restoring force directed towards the minimum point of the potential. If the object is compressed with  $x < 0$  then  $dU(x)/dx < 0$ ; the component of

the force is positive,  $F_x = -dU(x)/dx > 0$ , and the object again experiences a force back towards the minimum of the potential energy (Figure D.4.2).



**Figure D.4.2** Stability Diagram

Suppose our spring-object system has no loss of mechanical energy due to dissipative forces such as friction or air resistance. The energy at any time is the sum of the kinetic energy  $K(x)$  and the potential energy  $U(x)$

$$E = K(x) + U(x) \quad (\text{D.4.6})$$

Both the kinetic energy and the potential energy are functions of the position of the object with respect to equilibrium. The energy is a constant of the motion and with our choice of  $U(x_0 = 0) \equiv 0$  the energy can be positive or zero. When the energy is zero, the object is at rest at its equilibrium position. In Figure 5, we draw a straight line corresponding to a positive value for the energy on the graph of potential energy as a function of  $x$ .

The line corresponding to the energy intersects the potential energy function at two points  $\{-x_{\max}, x_{\max}\}$  with  $x_{\max} > 0$ . These points correspond to the maximum compression and maximum extension of the spring. They are called the “classical turning points.”

The kinetic energy is the difference between the energy and the potential energy,

$$K(x) = E - U(x) \quad (\text{D.4.7})$$

at the turning points, where  $E = U(x)$ , the kinetic energy is zero. Regions where the kinetic energy is negative,  $x < -x_{\max}$  or  $x > x_{\max}$ , are called the classically forbidden regions. The object can never reach these regions classically. In quantum mechanics, there is a very small probability that the object can be found in the classically forbidden regions.