MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

8.02

Review E: Simple Harmonic Motion and Mechanical Energy

This Worked Example demonstrates the basics of **Simple Harmonic Motion (SHO)** and so is included as a review unit.

An object of mass $m = 4.0\times10^{-2}\ \text{kg}$ sitting on a frictionless surface is attached to one end of a spring. The other end of the spring is attached to a wall. Assume that the object is constrained to move horizontally along one dimension. The spring has spring constant $k = 2.0\times10^{2}\ \text{N/m}$. The spring is initially stretched a distance $2.0\,\text{cm}$ from the equilibrium position and released at rest.

a) What is the position of the mass as a function of time?

b) What is the velocity of the mass as a function of time?

c) What is the time that it takes the mass-spring system to first return to its original configuration?

d) How do the initial conditions for the position and velocity of the mass-spring system enter into the solution?

e) What is the kinetic energy of the mass as a function of time?

f) What is the potential energy of the spring-mass system as a function of time?

g) What is mechanical energy of the spring-mass system as a function of time?

Solutions:

a) Choose the origin at the equilibrium position. Choose the positive x-direction to the right. Define $x(t)$ to be the position of the mass with respect to the equilibrium position.

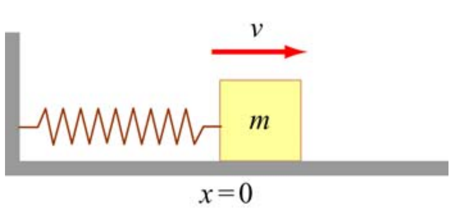

Figure 1 Mass-spring system

Newton's Second law in the horizontal direction $F_x = m a_x$ becomes

$$-k\,x = m\,a_x = m\frac{d^2x}{dt^2} \tag{E.1}$$

This equation is called the ***simple harmonic oscillator*** **(SHO)** equation. Since the spring force depends on the distance x, the acceleration is not constant. This is a second-order linear differential equation in which the second derivative in time of the position of the mass is proportional to the negative of the position of the mass,

$$\frac{d^2x}{dt^2} \propto -x \tag{E.2}$$

The constant of proportionality is k/m,

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \tag{E.3}$$

This equation can be solved directly by more advanced techniques, involving conservation of energy to obtain the speed v_x as a function of position x and "separation of variables." There is an existence and uniqueness theorem from the theory of differential equations which states that a unique solution exists which satisfies a given set of initial conditions $x_0 \equiv x(t=0)$ and $v_0 \equiv v(t=0)$ where x_0 and v_0 are constants. A second approach is to guess the solution and then verify that the guess satisfies the SHO differential equation. The guess for the solution takes the form

$$x(t) = A\cos(\omega t) + B\sin(\omega t) \tag{E.4}$$

The term ω is called the **angular frequency** (unfortunately the same symbol is used for angular velocity in circular motion but it should be clear that for a mass-spring system there is no circular motion). In order for the guess to satisfy the SHO equation, the angular frequency must satisfy

$$\omega = \sqrt{k/m} \tag{E.5}$$

Proof: To verify the guess, take the first and second derivatives of the guess and substitute the second derivative into the SHO equation,

$$dx/dt = -\omega\,A\sin(\omega t) + \omega\,B\cos(\omega t) \tag{E.6}$$

$$d^2x/dt^2 = -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t) = -\omega^2\left(A\cos(\omega t) + B\sin(\omega t)\right) = -\omega^2 x(t) \tag{E.7}$$

The last equality follows from substituting in our guess, $x(t) = A\cos(\omega t) + B\sin(\omega t)$. Thus the SHO equation becomes

$$\frac{d^2x}{dt^2} = -\omega^2 x = -\frac{k}{m}x \qquad \text{(E.8)}$$

This is satisfied providing

$$\omega = \sqrt{k/m} \qquad \text{(E.9)}$$

QED

The graph of position $x(t)$ *vs*. time t described by our solution is shown in Figure 2 .

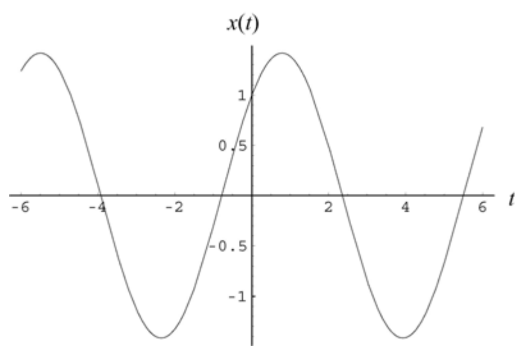

Figure 2 Graph of position $x(t)$ *vs*. time (with $A = B = 1$ and $\omega = 1$ rad/s .)

b) The velocity of the object at time t is then obtained by differentiating the solution,

$$v(t) = dx/dt = -\omega A\sin(\omega t) + \omega B\cos(\omega t) \qquad \text{(E.10)}$$

The graph of velocity $v(t)$ *vs*. time t is shown in Figure 3.

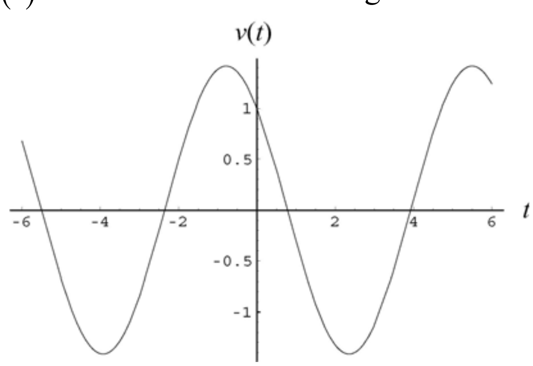

Figure 3 graph of velocity $v(t)$ *vs*. time (with $A = B = 1$ and $\omega = 1$ rad/s .)

c) The mass-spring system oscillates and returns back to its initial configuration for the first time at a time $t = T$ where the T is called the *period* and is defined by the condition

$$\omega T = 2\pi \tag{E.11}$$

since $\cos(0) = \cos(2\pi) = 1$, and $\sin(0) = \sin(2\pi) = 0$

Therefore the period is

$$T = 2\pi/\omega = 2\pi\big/\sqrt{k/m} = 2\pi\sqrt{k/m} \tag{E.12}$$

d) The guess for the solution takes the form

$$x(t) = A\cos(\omega t) + B\sin(\omega t) \tag{E.13}$$

where A and B are constants determined by the specific initial conditions

$$A = x_0 \text{ and } B = \frac{v_0}{\omega} \tag{E.14}$$

Proof: To find the constants A and B, substitute $t = 0$ into the guess for the solution. Since $\cos(0) = 1$ and $\sin(0) = 0$, the initial position at time $t = 0$ is

$$x_0 \equiv x(t = 0) = A \tag{E.15}$$

The velocity at time $t = 0$ is

$$v_0 = v(t = 0) = -\omega A\sin(0) + \omega B\cos(0) = \omega B \tag{E.16}$$

Thus

$$A = x_0 \text{ and } B = \frac{v_0}{\omega} \tag{E.17}$$

QED

Then the position of the spring-mass system is

$$x(t) = x_0\cos\left(\sqrt{k/m}\ t\right) + \frac{v_0}{\sqrt{k/m}}\sin\left(\sqrt{k/m}\ t\right) \tag{E.18}$$

and the velocity of the spring-mass system is

$$v(t) = -\sqrt{k/m}\, x_0 \sin\left(\sqrt{k/m}\; t\right) + v_0 \cos\left(\sqrt{k/m}\, t\right) \tag{E.19}$$

In the example, the angular frequency is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2.0\times10^{2}\ \text{N/m}}{4.0\times10^{-2}\ \text{kg}}} = 7.1\times10^{1}\ \text{rad/s} \tag{E.20}$$

$$A = x_0 = 2.0\times10^{-2}\ \text{m} \tag{E.21}$$

$$B = \frac{v_0}{\omega} = 0 \tag{E.22}$$

So the position is

$$x(t) = x_0 \cos\left(\sqrt{k/m}\; t\right) \tag{E.23}$$

and the velocity is

$$v(t) = -\sqrt{k/m}\, x_0 \sin\left(\sqrt{k/m}\; t\right) \tag{E.24}$$

e) The kinetic energy is

$$\frac{1}{2} m v^2 = \frac{1}{2} k\, x_0^2 \sin^2\left(\sqrt{k/m}\; t\right) \tag{E.25}$$

f) The potential energy is

$$\frac{1}{2} k\, x^2 = \frac{1}{2} k\, x_0^2 \cos^2\left(\sqrt{k/m}\; t\right) \tag{E.26}$$

g) The total mechanical energy is

$$E = K + U = \frac{1}{2} m v^2 + \frac{1}{2} k\, x^2 = \frac{1}{2} k\, x_0^2 \left(\cos^2\left(\sqrt{k/m}\; t\right) + \sin^2\left(\sqrt{k/m}\; t\right)\right) = \frac{1}{2} k\, x_0^2 \tag{E.27}$$

Since x_0 is a constant, the total energy is constant!

For our example the total energy is

$$E = \frac{1}{2} k\, x_0^2 = \frac{1}{2}\left(2.0\times10^{2}\ \text{N/m}\right)\left(2.0\times10^{-2}\ \text{m}\right)^2 = \left(2.0\times10^{2}\ \text{J}\right) \tag{E.28}$$