Section 1.1 : The Binomial Coefficients

Definition 1.1.1: Let n be a positive integer (i.e. a positive whole number). Then the *factorial* of n, denoted $n!$ and called "n factorial" is defined by

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$$

e.g.

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120, \quad 3! = 3 \times 2 \times 1 = 6, \ (etc.)$$

Note: $0!$ is defined to be 1.

Definition 1.1.2: Let n and k be non-negative integers with $k \leq n$. Then the *binomial coefficient* $\binom{n}{k}$ is defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Note: $\binom{n}{k}$ is called "n choose k": it is the number of ways of choosing k objects from a set of n objects.

Example 1.1.3*: $\displaystyle\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{(2 \times 1) \times (3 \times 2 \times 1)} = 10.$

So there are ten ways of choosing two objects from a set of 5. If $S = \{a, b, c, d, e\}$, then the two-element subsets of S are listed below :

$$\{a, b\}, \ \{a, c\}, \ \{a, d\}, \ \{a, e\}, \ \{b, c\}, \ \{b, d\}, \ \{b, e\}, \ \{c, d\}, \ \{c, e\}, \ \{d, e\}.$$

Example 1.1.4*: The number of ways of selecting a 5-member basketball team from a panel of 12 players is

$$\binom{12}{5} = \frac{12!}{5!(12-5)!} = \frac{12!}{5!(7)!} = \frac{12 \times 11 \times 10 \times 9 \times 8}{5 \times 4 \times 3 \times 2 \times 1} = 792.$$

Properties of Binomial Coefficients

1. $\binom{n}{0} = \binom{n}{n} = 1$ for all non-negative integers n. (A "non-negative" integer may be zero, or positive).
2. $\binom{n}{k} = \binom{n}{n-k}$ for all non-negative integers n, k with $k \leq n$.
3. $\binom{n}{1} = n$ for all non-negative integers n.

Pascal's Triangle : an interesting property of binomial coefficients

A triangular array of numbers can be constructed as follows :-

1. The first row, labelled Row 0, has a single entry: 1
2. Subsequently, Row n has $n+1$ entries, of which the first and last are both 1. Each other entry is the sum of the two above it, to the left and to the right.

$$\begin{array}{lccccccccccc}
\text{Row } 0 & & & & & & 1 & & & & & \\
\text{Row } 1 & & & & & 1 & & 1 & & & & \\
\text{Row } 2 & & & & 1 & & 2 & & 1 & & & \\
\text{Row } 3 & & & 1 & & 3 & & 3 & & 1 & & \\
\text{Row } 4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
\text{Row } 5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

This array is called *Pascal's Triangle*. The $n+1$ entries in Row n of the triangle are :

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}.$$

$$\begin{array}{lccccccccccc}
\text{Row } 0 & & & & & & \binom{0}{0} & & & & & \\
\text{Row } 1 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
\text{Row } 2 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
\text{Row } 3 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
\text{Row } 4 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\text{Row } 5 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}$$

So, for example, the six numbers in Row 5 of Pascal's triangle are $\binom{5}{0}$, $\binom{5}{1}, \ldots, \binom{5}{5}$.

Exercises :

1. Determine $\binom{7}{4}$ using either Pascal's triangle or Definition 1.1.2 (or both). (Answer is 35).
2. In how many ways can a 4-member committee be selected from a group of 10 candidates? (Answer is 210).