Section 1.2 : The Binomial Theorem

What happens if we take successive positive integer powers of the expresion "$x + y$"?

$$
\begin{aligned}
(x+y)^2 &= x^2 + 2xy + y^2 \\
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\
(x+y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4
\end{aligned}
$$

In general $(x + y)^n$ is a sum of terms such as $x^n,\ x^{n-1}y,\ x^{n-1}y^2, \ldots, x^1y^{n-1}, y^n$, with some integer coefficients. The *Binomial Theorem* tells us exactly what these coefficients are.

Theorem 1.2.1: (Binomial Theorem) Let n be a positive integer. Then

$$(x+y)^n = \binom{n}{0}x^ny^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1y^{n-1} + \binom{n}{n}x^0y^n.$$

Example 1.2.2*: If $n = 3$:

$$
\begin{aligned}
(x+y)^3 &= \binom{3}{0}x^3y^0 + \binom{3}{1}x^2y^1 + \binom{3}{2}x^1y^2 + \binom{3}{3}x^0y^3 \\
&= x^3 + 3x^2y + 3xy^2 + y^3
\end{aligned}
$$

Example 1.2.3*: Use the binomial theorem to write $(x+2)^5$ as a single polynomial.

$$(x+2)^5 = \binom{5}{0}x^52^0 + \binom{5}{1}x^42^1 + \binom{5}{2}x^32^2 + \binom{5}{3}x^22^3 + \binom{5}{4}x^12^4 + \binom{5}{5}x^02^5$$

For the values of the binomial coefficients look at Row 5 of Pascal's Triangle.

$$
\begin{aligned}
(x+2)^5 &= 1x^5(1) + 5x^4(2) + 10x^3(4) + 10x^2(8) + 5x(16) + 1(32) \\
&= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32
\end{aligned}
$$

Example 1.2.4*: What is the coefficient of x^3 in the expansion of $(x - 3)^8$?

Solution: $(x-3)^8 = \binom{8}{0}x^8 + \binom{8}{1}x^7(-3)^1 + \cdots + \boldsymbol{\binom{8}{5}x^3(-3)^5} + \ldots$

So the coefficient of x^3 is

$$
\begin{aligned}
\binom{8}{5}(-3)^5 &= \frac{8!}{5!(8-5)!}(-243) \\
&= 56 \times (-243) \\
&= 13608
\end{aligned}
$$

Exercise : What is the coefficient of x^5 in $(x + 2)^9$? (Answer 1008)