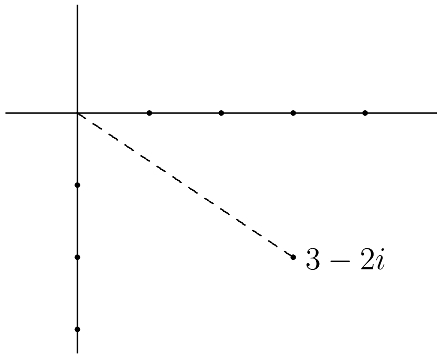
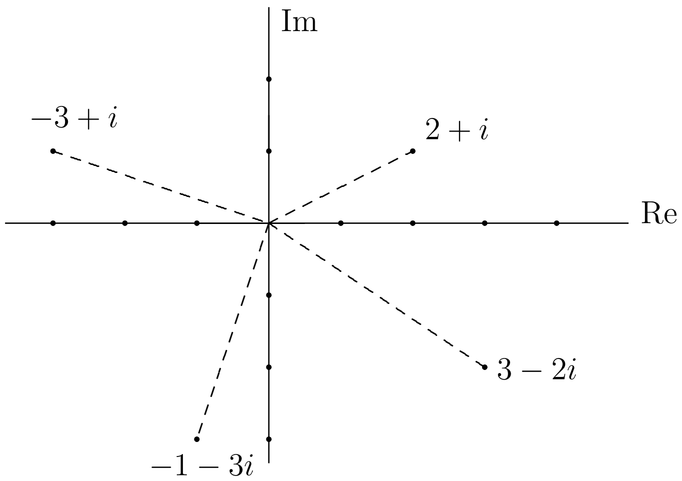
Section 8.2 : The Polar Form of a Complex Number

A complex number such as $3 - 2i$ may be represented graphically as the point in 2-dimensional space with coordinates $(3, -2)$ (i.e. (*real part*, *imaginary part*)).



In this respect a complex number resembles a vector or point in $\mathbb{R}^2$. In this context the horizontal (X) and vertical (Y) axes are called the real and imaginary axes respectively and denoted Re and Im. A diagram depicting complex numbers like this is called an *Argand diagram.*



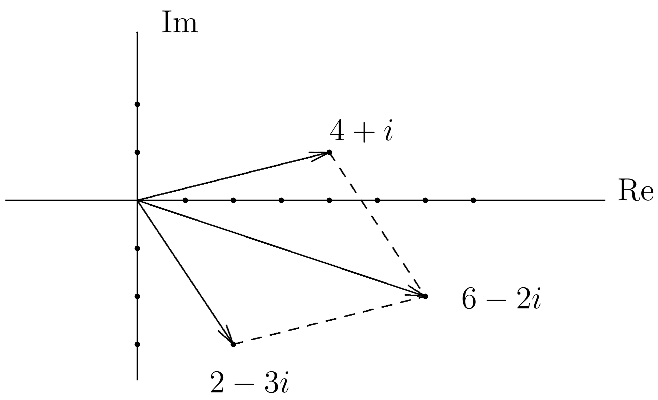
Argand Diagram

<u>Remarks</u>

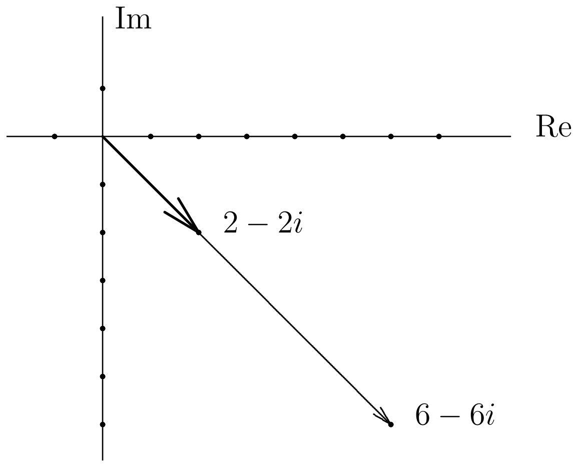
1. The definition of addition of complex numbers is consistent with vector addition in $\mathbb{R}^2$: for example

$$(2 - 3i) + (4 + i) = 6 - 2i \text{ in } \mathbb{C}$$

In $\mathbb{R}^2$, the sum of the vectors $[2, -3]$ and $[4, 1]$ is $[6, -2]$.



2. If c is a real number and z is a complex number, then the vector representing cz is just "$c\times$(vector representing z)". For example $3 \times (2 - 2i) = 6 - 6i$ in $\mathbb{C}$. Correspondingly, $3[2, -2] = [6, -6]$ in $\mathbb{R}^2$.



Definition 8.2.1: If z is a complex number, the *modulus* or length of z, denoted $|z|$, is defined as the length of the vector which represents z.

Example: If $z = 3 + 4i$ then $|z| = ||[3, 4]|| = \sqrt{3^2 + 4^2} = 5$.

In general if $z = a + bi$ then

$$|z| = ||[a, b]|| = \sqrt{a^2 + b^2}$$

e.g. $|2 - 2i| = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$

Remarks

1. For every complex number z, $|z|$ is a non-negative real number. Also, $|z| = 0$ if and only if $z = 0\,(= 0 + 0i)$.

2. If $z = a + bi$ then $z\bar{z} = (a + bi)(a - bi) = a^2 + b^2 = |z|^2$.

3. If z is a complex number and c is a real number then $|cz| = |c| \times |z|$.

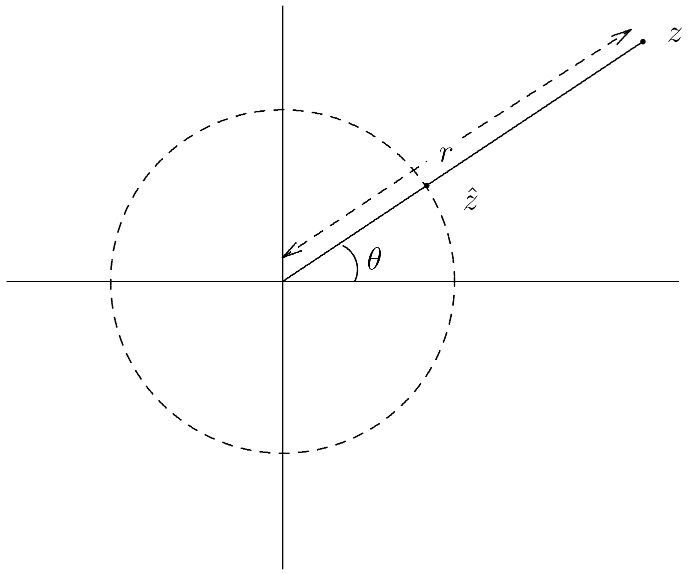
Let z be a non-zero complex number of modulus r. So $r > 0$. Define

$$\hat{z} = \frac{1}{r}z$$

Then $\hat{z}$ has modulus 1.

$\Longrightarrow$ the point representing $\hat{z}$ lies on the unit circle.

$\Longrightarrow$ this point has coordinates $(\cos\theta, \sin\theta)$, where θ is the angle between the positive real axis and the vector representing z.



Thus

$$\hat{z} = \frac{1}{r}z = \cos\theta + i\sin\theta$$

and

$$z = r(\cos\theta + i\sin\theta)$$

Definition 8.2.2: The angle θ between the positive real axis and the vector representing z is called the *argument* of the complex number z and is denoted $\arg(z)$. It is conventional to choose $\arg(z)$ between $-\pi$ and π.

We have shown: If z is a complex number with modulus r and argument θ then

$$z = r(\cos\theta + i\sin\theta)$$

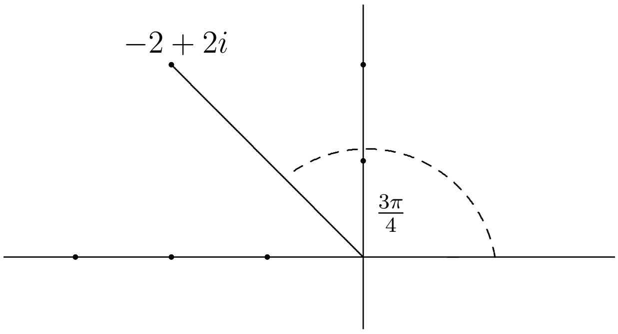
or

$$z = r\cos\theta + i\,r\sin\theta$$

A complex number written in this way in terms of its modulus and argument is said to be in *polar form*.

Example 8.2.3*: Write $-2+2i$ in polar form.

Solution: We need the modulus and argument of z



Modulus: $|z| = \sqrt{(-2)^2+(2)^2} = \sqrt{8} = 2\sqrt{2}$

Argument: To find $\theta = \arg(z)$: First note that θ is in the 2nd quadrant. Then

$$\tan\theta = \frac{\text{Imaginary Part}}{\text{Real Part}} = \frac{2}{-2} = -1$$

Hence $\theta = \frac{3\pi}{4}$, and

$$z = -2+2i = 2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$$