Section 8.3 : De Moivre's Theorem and Applications

Let z_1 and z_2 be complex numbers, where

$$|z_1| = r_1,\ |z_2| = r_2,\ \arg(z_1) = \theta_1,\ \arg(z_2) = \theta_2.$$

Then

$$\begin{aligned} z_1 &= r_1(\cos\theta_1 + i\,\sin\theta_1) \\ z_2 &= r_2(\cos\theta_2 + i\,\sin\theta_2) \end{aligned}$$

and

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \left(\underbrace{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)}_{\cos(\theta_1+\theta_2)} + i\underbrace{(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)}_{\sin(\theta_1+\theta_2)} \right) \\ &= r_1 r_2 \left(\cos(\theta_1+\theta_2) + i\,\sin(\theta_1+\theta_2)\right) \end{aligned}$$

This means

1. $|z_1 z_2| = r_1 r_2 = |z_1|\,|z_2|$
2. $\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

or :

The modulus of the product of two complex numbers is the product of their moduli,
and
the argument of the product of two complex numbers is the sum of their arguments.

We can use these facts to compute the square of a complex number (in polar form): suppose $z = r(\cos\theta + i\,\sin\theta)$, so $|z| = r$ and $\arg(z) = \theta$. Then z^2 has modulus $r \times r = r^2$, and z^2 has argument $\theta + \theta = 2\theta$, i.e.

$$z^2 = r^2(\cos(2\theta) + i\,\sin(2\theta))$$

This principle can be used to compute any positive integer power of z to give :

Theorem 8.3.1: (De Moivre's Theorem) Let $z = r(\cos\theta + i\,\sin\theta)$, and let n be a positive integer. Then

$$z^n = r^n(\cos n\theta + i\,\sin n\theta)$$

(i.e. in taking the nth power of z, we raise the modulus to its nth power and multiply the argument by n.)

Remark: Provided $z \neq 0$, De Moivre's Theorem also holds for negative integers n.

We now consider three problems of different types, all involving De Moivre's theorem.

1. Computing Positive Powers of a Complex Number

Example 8.3.2* Let $z = 1 - i$. Find z^{10}.

Solution: First write z in polar form.

$|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\arg(z) = -\frac{\pi}{4}$ (or $\frac{7\pi}{4}$)

Polar Form : $z = \sqrt{2}\left(\cos(-\frac{\pi}{4}) + i\,\sin(-\frac{\pi}{4})\right)$.

Applying de Moivre's Theorem gives :

$$\begin{aligned} z^{10} &= (\sqrt{2})^{10}\left(\cos(10 \times (-\frac{\pi}{4})) + i\,\sin(10 \times (-\frac{\pi}{4}))\right) \\ &= 2^5\left(\cos(-\frac{10\pi}{4}) + i\,\sin(-\frac{10\pi}{4})\right) \\ &= 32\left(\cos(-\frac{5\pi}{2}) + i\,\sin(-\frac{5\pi}{2})\right) \\ &= 32\left(\cos(-\frac{5\pi}{2} + 2\pi) + i\,\sin(-\frac{5\pi}{2} + 2\pi)\right) \\ &= 32\left(\cos(-\frac{\pi}{2}) + i\,\sin(-\frac{\pi}{2})\right) \\ &= 32(0 + i\,(-1)) \\ &= -32i \end{aligned}$$

Note: It can be verified directly that $(1 - i)^{10} = -32i$.

Exercise : Use De Moivre's Theorem to find $(1 + \sqrt{3}i)^6$.

2. Computing nth roots of a complex number.

Example 8.3.3* Find all complex cube roots of $27i$.

Solution: We are looking for complex numbers z with the property $z^3 = 27i$.

Strategy: First we write $27i$ in polar form :-

$|27i| = |0 + 27i| = \sqrt{0^2 + (27)^2} = 27$

$\arg(27i) = \frac{\pi}{2}$

$$27i = 27(\cos\frac{\pi}{2} + i\,\sin\frac{\pi}{2})$$

Now suppose $z = r(\cos\theta + i\,\sin\theta)$ satisfies $z^3 = 27i$. Then, by De Moivre's Theorem,

$$r^3(\cos\,3\theta + i\,\sin\,3\theta) = 27i = 27(\cos\frac{\pi}{2} + i\,\sin\frac{\pi}{2})$$

Thus $r^3 = 27 \Longrightarrow r = 3$ (since r must be a positive real number with cube 27).

What are the possible values of θ? We must have

$$\cos\,3\theta = \cos\frac{\pi}{2} \text{ and } \sin\,3\theta = \sin\frac{\pi}{2}$$

This means :

$$3\theta = \frac{\pi}{2} + 2\pi k,$$

where k is an integer; i.e. 3θ differs from $\frac{\pi}{2}$ by a multiple of 2π. Possibilities are :

1. $k = 0$: $3\theta = \frac{\pi}{2}$, $\theta = \frac{\pi}{6}$

$$\begin{aligned} z_1 &= 3(\cos\frac{\pi}{6} + i\,\sin\frac{\pi}{6}) \\ &= 3(\frac{\sqrt{3}}{2} + i\,\frac{1}{2}) \\ z_1 &= \frac{3\sqrt{3}}{2} + \frac{3}{2}i \end{aligned}$$

2. $k = 1$: $3\theta = \frac{\pi}{2} + 2\pi(1) = \frac{5\pi}{2}$, $\theta = \frac{5\pi}{6}$

$$\begin{aligned} z_2 &= 3(\cos\frac{5\pi}{6} + i\,\sin\frac{5\pi}{6}) \\ &= 3(-\frac{\sqrt{3}}{2} + i\,\frac{1}{2}) \\ z_2 &= -\frac{3\sqrt{3}}{2} + \frac{3}{2}i \end{aligned}$$

3. $k = 2$: $3\theta = \frac{\pi}{2} + 2\pi(2) = \frac{9\pi}{2}$, $\theta = \frac{9\pi}{6} = \frac{3\pi}{2}$

$$\begin{aligned} z_3 &= 3(\cos\frac{3\pi}{2} + i\,\sin\frac{3\pi}{2}) \\ &= 3(0 + i(-1)) \\ z_1 &= -3i \end{aligned}$$

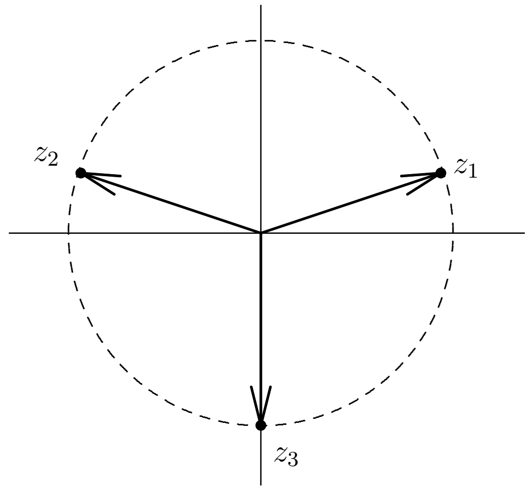
These are the only possibilities : setting $k = 3$ results in $\theta = \frac{\pi}{2} + 2\pi$ which gives the same result as $k = 0$.

The complex cube roots of $-27i$ are :

$z_1 = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$

$z_2 = \frac{-3\sqrt{3}}{2} + \frac{3}{2}i$

$z_3 = -3i$



In general : To find the complex nth roots of a non-zero complex number z.

1. Write z in polar form : $z = r(\cos\theta + i\,\sin\theta)$
2. z will have n different nth roots (i.e. 3 cube roots, 4 fourth roots, etc.).
3. All these roots will have the same modulus $r^{\frac{1}{n}}$ (the positive real nth root of r).
4. They will have different arguments :

$$\frac{\theta}{n}, \quad \frac{\theta + 2\pi}{n}, \quad \frac{\theta + (2 \times 2\pi)}{n}, \quad \dots \quad , \frac{\theta + ((n-1) \times 2\pi)}{n}$$

5. The complex nth roots of z are given (in polar form) by

$$\begin{aligned} z_1 &= r^{\frac{1}{n}} \left(\cos(\tfrac{\theta}{n}) + i\,\sin(\tfrac{\theta}{n})\right) \\ z_2 &= r^{\frac{1}{n}} \left(\cos(\tfrac{\theta+2\pi}{n}) + i\,\sin(\tfrac{\theta+2\pi}{n})\right) \\ z_3 &= r^{\frac{1}{n}} \left(\cos(\tfrac{\theta+4\pi}{n}) + i\,\sin(\tfrac{\theta+4\pi}{n})\right), \text{ etc.} \end{aligned}$$

Example: Find all the complex fourth roots of -16.

Solution: First write -16 in polar form.

Modulus : 16

Argument : π

$$-16 = 16(\cos\pi + i\,\sin\pi)$$

Fourth roots of 16 all have modulus $16^{\frac{1}{4}} = 2$, and possibilities for the argument are :

$$\frac{\pi}{4},\quad \frac{\pi+2\pi}{4} = \frac{3\pi}{4},\quad \frac{\pi+4\pi}{4} = \frac{5\pi}{4},\quad \frac{\pi+3\pi}{4} = \frac{7\pi}{4}$$

Fourth roots of -16 are :-

$$\begin{aligned} z_1 &= 2(\cos(\tfrac{\pi}{4}) + i\,\sin(\tfrac{\pi}{4})) = \sqrt{2} + \sqrt{2}i \\ z_2 &= 2(\cos(\tfrac{3\pi}{4}) + i\,\sin(\tfrac{3\pi}{4})) = -\sqrt{2} + \sqrt{2}i \\ z_3 &= 2(\cos(\tfrac{5\pi}{4}) + i\,\sin(\tfrac{5\pi}{4})) = -\sqrt{2} - \sqrt{2}i \\ z_4 &= 2(\cos(\tfrac{7\pi}{4}) + i\,\sin(\tfrac{7\pi}{4})) = \sqrt{2} - \sqrt{2}i \end{aligned}$$

3. Proving Trigonometric Identities

Example 8.3.4*: Prove that

1. $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$
2. $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

Solution: The idea is to write $(\cos\theta + i\,\sin\theta)^5$ in two different ways. We use both the binomial theorem and De Moivre's theorem, and compare the results.

Binomial Theorem:

$(\cos\theta + i\,\sin\theta)^5 =$

$$\begin{aligned} & (\cos\theta)^5 + \binom{5}{1}(\cos\theta)^4(i\,\sin\theta)^1 + \binom{5}{2}(\cos\theta)^3(i\,\sin\theta)^2 + \binom{5}{3}(\cos\theta)^2(i\,\sin\theta)^3 \\ & + \binom{5}{4}(\cos\theta)^1(i\,\sin\theta)^4 + \binom{5}{5}(\cos\theta)^0(i\sin\theta)^5 \\ =\; & \cos^5\theta + 5\cos^4\theta(i\,\sin\theta) + 10(\cos^3\theta)(i^2\,\sin^2\theta) + 10(\cos^2\theta)(i^3\,\sin^3\theta) \\ & \quad +5(\cos\theta)(i^4\,\sin^4\theta) + (i^5\sin^5\theta) \\ =\; & \cos^5\theta + i\,5\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - i\,10\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\,\sin^5\theta \\ =\; & (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta) \end{aligned}$$

Also, by De Moivre's Theorem, we have

$$(\cos\theta + i\,\sin\theta)^5 = \cos 5\theta + i\,\sin 5\theta$$

and so

$$\begin{aligned}\cos 5\theta + i\,\sin 5\theta &= (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) \\ &\quad + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)\end{aligned}$$

Equating the real parts gives

$$\begin{aligned}\cos 5\theta &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\ &= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2 \\ &= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta(1-2\cos^2\theta+\cos^4\theta) \\ &= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta \\ \cos 5\theta &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\end{aligned}$$

For the other identity, look at the imaginary parts :

$$\begin{aligned}\sin 5\theta &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta \\ &= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta \\ &= 5(1-2\sin^2\theta+\sin^4\theta)\sin\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta \\ &= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta \\ &= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta \\ \sin 5\theta &= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta\end{aligned}$$

Remark: This method can be used to prove many trigonometric identities. In general one can write $\sin n\theta$ and $\cos n\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$ by using both the binomial theorem and De Moivre's theorem to expand $(\cos\theta + i\,\sin\theta)^n$ and comparing the real and imaginary parts of the results.

Exercise: Prove :

1. $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$
2. $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$