32. Describe the set of points in the complex plane that satisfy the following.

(a) $|z| = 4$ (b) $|z - i| = 2$

(c) $|z + 1| \leq 1$ (d) $|z| > 3$

33. (a) Evaluate $(1/i)^n$ for $n = 1, 2, 3, 4$, and 5.

(b) Calculate $(1/i)^{57}$ and $(1/i)^{1995}$.

(c) Find a general formula for $(1/i)^n$ for any positive integer n.

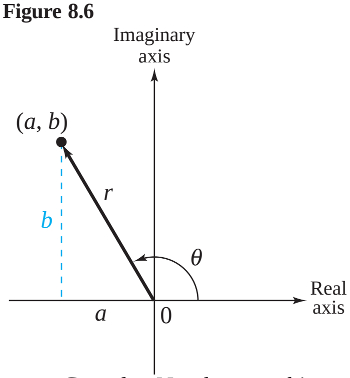
34. (a) Verify that $\left(\frac{1 + i}{\sqrt{2}}\right)^2 = i$.

(b) Find the two square roots of i.

(c) Find all zeros of the polynomial $x^4 + 1$.

8.3 POLAR FORM AND DEMOIVRE'S THEOREM

Figure 8.6



Complex Number: $a + bi$
Rectangular Form: (a, b)
Polar Form: (r, θ)

At this point we can add, subtract, multiply, and divide complex numbers. However, there is still one basic procedure that is missing from our algebra of complex numbers. To see this, consider the problem of finding the square root of a complex number such as i. When we use the four basic operations (addition, subtraction, multiplication, and division), there seems to be no reason to guess that

$$\sqrt{i} = \frac{1 + i}{\sqrt{2}}. \quad \text{That is,} \quad \left(\frac{1 + i}{\sqrt{2}}\right)^2 = i.$$

To work effectively with *powers* and *roots* of complex numbers, it is helpful to use a polar representation for complex numbers, as shown in Figure 8.6. Specifically, if $a + bi$ is a nonzero complex number, then we let θ be the angle from the positive x-axis to the radial line passing through the point (a, b) and we let r be the modulus of $a + bi$. Thus,

$$a = r\cos\theta, \qquad b = r\sin\theta, \qquad \text{and} \qquad r = \sqrt{a^2 + b^2}$$

and we have $a + bi = (r\cos\theta) + (r\sin\theta)i$ from which we obtain the following **polar form** of a complex number.

Definition of Polar Form of a Complex Number

The **polar form** of the nonzero complex number $z = a + bi$ is given by

$$z = r(\cos\theta + i\sin\theta)$$

where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$, and $\tan\theta = b/a$. The number r is the **modulus** of z and θ is called the **argument** of z.

REMARK: The polar form of $z = 0$ is given by $z = 0(\cos\theta + i\sin\theta)$ where θ is any angle.

Because there are infinitely many choices for the argument, the polar form of a complex number is not unique. Normally, we use values of θ that lie between $-\pi$ and π, though on occasion it is convenient to use other values. The value of θ that satisfies the inequality

$$-\pi < \theta \leq \pi \qquad \text{Principal argument}$$

is called the **principal argument** and is denoted by Arg(z). Two nonzero complex numbers in polar form are equal if and only if they have the same modulus and the same principal argument.

EXAMPLE 1 ***Finding the Polar Form of a Complex Number***

Find the polar form of the following complex numbers. (Use the principal argument.)

(a) $1 - i$ (b) $2 + 3i$ (c) i

Solution

(a) We have $a = 1$ and $b = -1$, so $r^2 = 1^2 + (-1)^2 = 2$, which implies that $r = \sqrt{2}$. From $a = r\cos\theta$ and $b = r\sin\theta$, we have

$$\cos\theta = \frac{a}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin\theta = \frac{b}{r} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

Thus, $\theta = -\pi/4$ and

$$z = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right].$$

(b) Since $a = 2$ and $b = 3$, we have $r^2 = 2^2 + 3^2 = 13$, which implies that $r = \sqrt{13}$. Therefore,

$$\cos\theta = \frac{a}{r} = \frac{2}{\sqrt{13}} \quad \text{and} \quad \sin\theta = \frac{b}{r} = \frac{3}{\sqrt{13}}$$

and it follows that $\theta = \arctan(3/2)$. Therefore, the polar form is

$$z = \sqrt{13}\left[\cos\left(\arctan\frac{3}{2}\right) + i\sin\left(\arctan\frac{3}{2}\right)\right]$$

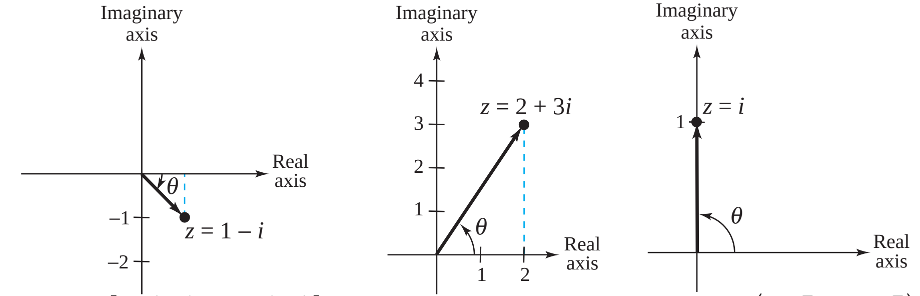
$$\approx \sqrt{13}[\cos(0.98) + i\sin(0.98)].$$

(c) Since $a = 0$ and $b = 1$, it follows that $r = 1$ and $\theta = \pi/2$, so we have

$$z = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right).$$

The polar forms derived in parts (a), (b), and (c) are depicted graphically in Figure 8.7.

Figure 8.7



(a) $z = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$ (b) $z = \sqrt{13}[\cos(0.98) + i\sin(0.98)]$ (c) $z = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

EXAMPLE 2 *Converting from Polar to Standard Form*

Express the following complex number in standard form.

$$z = 8\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$$

Solution Since $\cos(-\pi/3) = 1/2$ and $\sin(-\pi/3) = -\sqrt{3}/2$, we obtain the standard form

$$z = 8\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right] = 8\left[\frac{1}{2} - i\frac{\sqrt{3}}{2}\right] = 4 - 4\sqrt{3}i.$$

The polar form adapts nicely to multiplication and division of complex numbers. Suppose we are given two complex numbers in polar form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

Then the product of z_1 and z_2 is given by

$$\begin{aligned} z_1z_2 &= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)]. \end{aligned}$$

Using the trigonometric identities

$$\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$

and

$$\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$$

we have

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

This establishes the first part of the following theorem. The proof of the second part is left to you. (See Exercise 63.)

Theorem 8.4
Product and Quotient of Two Complex Numbers

Given two complex numbers in polar form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

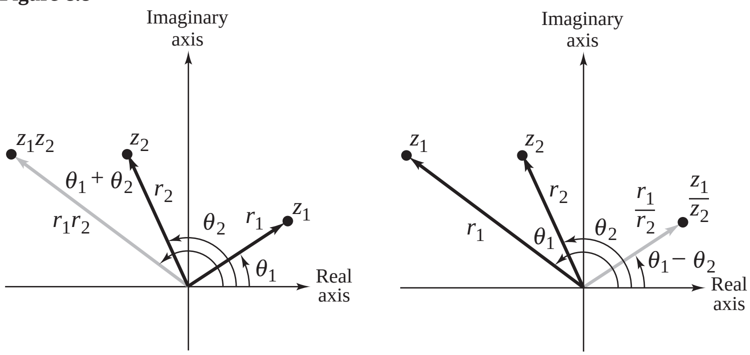
the product and quotient of the numbers are as follows.

$$z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)] \qquad \textbf{Product}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)],\ z_2 \neq 0 \qquad \textbf{Quotient}$$

This theorem says that to multiply two complex numbers in polar form, we multiply moduli and add arguments, and to divide two complex numbers, we divide moduli and subtract arguments. (See Figure 8.8.)

Figure 8.8



To multiply z_1 and z_2:
Multiply moduli and add arguments.

To divide z_1 by z_2:
Divide moduli and add arguments.

EXAMPLE 3 ***Multiplying and Dividing in Polar Form***

Determine z_1z_2 and z_1/z_2 for the complex numbers

$$z_1 = 5\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \quad \text{and} \quad z_2 = \frac{1}{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right).$$

Solution Since we are given the polar forms of z_1 and z_2, we can apply Theorem 8.4 as follows.

$$z_1z_2 = \overbrace{(5)\left(\frac{1}{3}\right)}^{\textit{multiply}}\left[\cos\underbrace{\left(\frac{\pi}{4}+\frac{\pi}{6}\right)}_{\textit{add}} + i\sin\underbrace{\left(\frac{\pi}{4}+\frac{\pi}{6}\right)}_{\textit{add}}\right] = \frac{5}{3}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$$

$$\frac{z_1}{z_2} = \overbrace{\frac{5}{1/3}}^{\textit{divide}}\left[\cos\underbrace{\left(\frac{\pi}{4}-\frac{\pi}{6}\right)}_{\textit{subtract}} + i\sin\underbrace{\left(\frac{\pi}{4}-\frac{\pi}{6}\right)}_{\textit{subtract}}\right] = 15\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

REMARK: Try performing the multiplication and division in Example 3 using the standard forms

$$z_1 = \frac{5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}i \quad \text{and} \quad z_2 = \frac{\sqrt{3}}{6} + \frac{1}{6}i.$$

DeMoivre's Theorem

Our final topic in this section involves procedures for finding powers and roots of complex numbers. Repeated use of multiplication in the polar form yields

$$\begin{aligned} z &= r(\cos\theta + i\sin\theta) \\ z^2 &= r(\cos\theta + i\sin\theta)\, r(\cos\theta + i\sin\theta) = r^2(\cos 2\theta + i\sin 2\theta) \\ z^3 &= r(\cos\theta + i\sin\theta)\, r^2(\cos 2\theta + i\sin 2\theta) = r^3(\cos 3\theta + i\sin 3\theta). \end{aligned}$$

Similarly,

$$\begin{aligned} z^4 &= r^4(\cos 4\theta + i\sin 4\theta) \\ z^5 &= r^5(\cos 5\theta + i\sin 5\theta). \end{aligned}$$

This pattern leads to the following important theorem, named after the French mathematician Abraham DeMoivre (1667–1754). You are asked to prove this theorem in Chapter Review Exercise 71.

Theorem 8.5
DeMoivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ and n is any positive integer, then

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

EXAMPLE 4 ***Raising a Complex Number to an Integer Power***

Find $(-1 + \sqrt{3}i)^{12}$ and write the result in standard form.

Solution We first convert to polar form. For $-1 + \sqrt{3}i$, we have

$$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2 \quad \text{and} \quad \tan\theta = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

which implies that $\theta = 2\pi/3$. Therefore,

$$-1 + \sqrt{3}i = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right).$$

By DeMoivre's Theorem, we have

$$\begin{aligned} (-1 + \sqrt{3}i)^{12} &= \left[2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]^{12} \\ &= 2^{12}\left[\cos\frac{12(2\pi)}{3} + i\sin\frac{12(2\pi)}{3}\right] \end{aligned}$$

$$= 4096(\cos 8\pi + i \sin 8\pi)$$
$$= 4096[1 + i(0)] = 4096.$$

Recall that a consequence of the Fundamental Theorem of Algebra is that a polynomial of degree n has n zeros in the complex number system. Hence, a polynomial like $p(x) = x^6 - 1$ has six zeros, and in this case we can find the six zeros by factoring and using the quadratic formula.

$$x^6 - 1 = (x^3 - 1)(x^3 + 1) = (x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1)$$

Consequently, the zeros are

$$x = \pm 1, \quad x = \frac{-1 \pm \sqrt{3}i}{2}, \quad \text{and} \quad x = \frac{1 \pm \sqrt{3}i}{2}.$$

Each of these numbers is called a sixth root of 1. In general, we define the nth root of a complex number as follows.

Definition of nth Root of a Complex Number

The complex number $w = a + bi$ is an **nth root** of the complex number z if

$$z = w^n = (a + bi)^n.$$

DeMoivre's Theorem is useful in determining roots of complex numbers. To see how this is done, let w be an nth root of z, where

$$w = s(\cos \beta + i \sin \beta) \quad \text{and} \quad z = r(\cos \theta + i \cos \theta).$$

Then, by DeMoivre's Theorem we have $w^n = s^n(\cos n\beta + i \sin n\beta)$ and since $w^n = z$, it follows that

$$s^n(\cos n\beta + i \sin n\beta) = r(\cos \theta + i \sin \theta).$$

Now, since the right and left sides of this equation represent equal complex numbers, we can equate moduli to obtain $s^n = r$ which implies that $s = \sqrt[n]{r}$ and equate principal arguments to conclude that θ and $n\beta$ must differ by a multiple of 2π. Note that r is a positive real number and hence $s = \sqrt[n]{r}$ is also a positive real number. Consequently, for some integer k, $n\beta = \theta + 2\pi k$, which implies that

$$\beta = \frac{\theta + 2\pi k}{n}.$$

Finally, substituting this value for β into the polar form of w, we obtain the result stated in the following theorem.

Theorem 8.6

***n*th Roots of a Complex Number**

For any positive integer n, the complex number

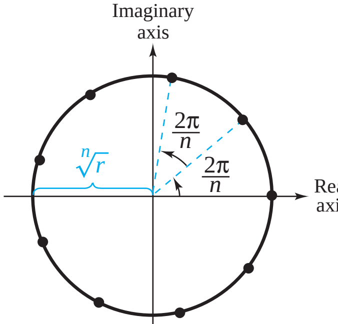
$$z = r(\cos\theta + i\sin\theta)$$

has exactly n distinct roots. These n roots are given by

$$\sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots, n - 1$.

Figure 8.9



*n*th Roots of a Complex Number

REMARK: Note that when k exceeds $n - 1$, the roots begin to repeat. For instance, if $k = n$, the angle is

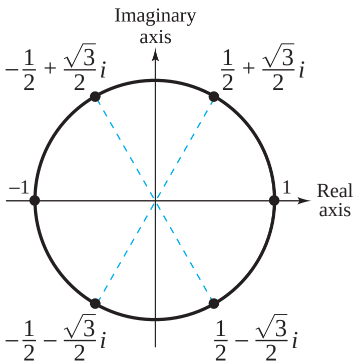
$$\frac{\theta + 2\pi n}{n} = \frac{\theta}{n} + 2\pi$$

which yields the same value for the sine and cosine as $k = 0$.

The formula for the *n*th roots of a complex number has a nice geometric interpretation, as shown in Figure 8.9. Note that because the *n*th roots all have the same modulus (length) $\sqrt[n]{r}$, they will lie on a circle of radius $\sqrt[n]{r}$ with center at the origin. Furthermore, the n roots are equally spaced along the circle, since successive *n*th roots have arguments that differ by $2\pi/n$.

We have already found the sixth roots of 1 by factoring and the quadratic formula. Try solving the same problem using Theorem 8.6 to see if you get the roots shown in Figure 8.10. When Theorem 8.6 is applied to the real number 1, we give the *n*th roots a special name—the ***n*th roots of unity.**

Figure 8.10



6th Roots of Unity

EXAMPLE 5 *Finding the nth Roots of a Complex Number*

Determine the fourth roots of i.

Solution In polar form, we can write i as

$$i = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

so that $r = 1$, $\theta = \pi/2$. Then, by applying Theorem 8.6, we have

$$i^{1/4} = \sqrt[4]{1}\left[\cos\left(\frac{\pi/2}{4} + \frac{2k\pi}{4}\right) + i\sin\left(\frac{\pi/2}{4} + \frac{2k\pi}{4}\right)\right]$$

$$= \cos\left(\frac{\pi}{8} + \frac{k\pi}{2}\right) + i\sin\left(\frac{\pi}{8} + \frac{k\pi}{2}\right).$$

Setting $k = 0, 1, 2$, and 3 we obtain the four roots

$$z_1 = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}$$

$$z_2 = \cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}$$

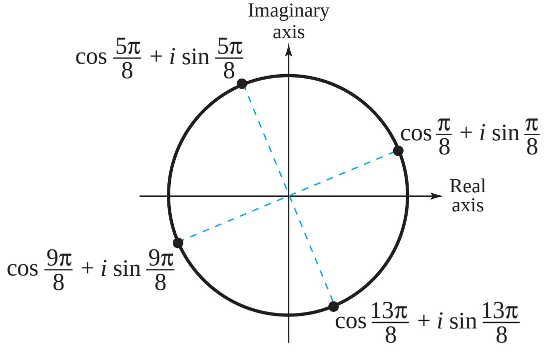
$$z_3 = \cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}$$

$$z_4 = \cos\frac{13\pi}{8} + i\sin\frac{13\pi}{8}$$

as shown in Figure 8.11.

REMARK: In Figure 8.11 note that when each of the four angles, $\pi/8, 5\pi/8, 9\pi/8$, and $13\pi/8$ is multiplied by 4, the result is of the form $(\pi/2) + 2k\pi$.

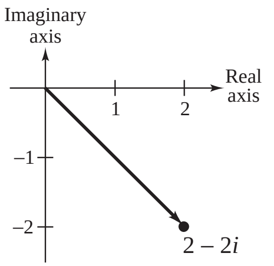
Figure 8.11



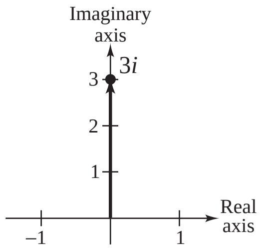
SECTION 8.3 ❑ EXERCISES

In Exercises 1–4, express the complex number in polar form.

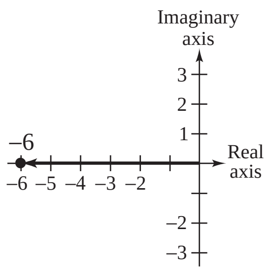
1.



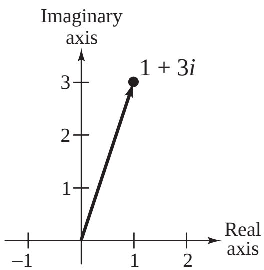
2.



3.



4.



In Exercises 5–16, represent the complex number graphically, and give the polar form of the number.

5. $-2 - 2i$ **6.** $\sqrt{3} + i$

7. $-2(1 + \sqrt{3}\,i)$ **8.** $\frac{5}{2}(\sqrt{3} - i)$

9. $6i$ **10.** 4

11. 7 **12.** $-2i$

13. $1 + 6i$ **14.** $2\sqrt{2} - i$

15. $-3 - i$ **16.** $-4 + 2i$

In Exercises 17–26, represent the complex number graphically, and give the standard form of the number.

17. $2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ **18.** $5\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

19. $\frac{3}{2}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$ **20.** $\frac{3}{4}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

21. $3.75\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ **22.** $8\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

23. $4\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$ **24.** $6\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

25. $7(\cos 0 + i\sin 0)$ **26.** $6(\cos\pi + i\sin\pi)$

In Exercises 27–34, perform the indicated operation and leave the result in polar form.

27. $\left[3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]\left[4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]$

28. $\left[\frac{3}{4}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\right]\left[6\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]$

29. $[0.5(\cos\pi + i\sin\pi)][0.5(\cos[-\pi] + i\sin[-\pi])]$

30. $\left[3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]\left[\frac{1}{3}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]$

31. $\dfrac{2[\cos(2\pi/3) + i\sin(2\pi/3)]}{4[\cos(2\pi/9) + i\sin(2\pi/9)]}$

32. $\dfrac{\cos(5\pi/3) + i\sin(5\pi/3)}{\cos\pi + i\sin\pi}$

33. $\dfrac{12[\cos(\pi/3) + i\sin(\pi/3)]}{3[\cos(\pi/6) + i\sin(\pi/6)]}$

34. $\dfrac{9[\cos(3\pi/4) + i\sin(3\pi/4)]}{5[\cos(-\pi/4) + i\sin(-\pi/4)]}$

In Exercises 35–44, use DeMoivre's Theorem to find the indicated powers of the given complex number. Express the result in standard form.

35. $(1 + i)^4$ **36.** $(2 + 2i)^6$

37. $(-1 + i)^{10}$ **38.** $(\sqrt{3} + i)^7$

39. $(1 - \sqrt{3}i)^3$ **40.** $\left[5\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)\right]^3$

41. $\left[3\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)\right]^4$ **42.** $\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)^{10}$

43. $\left[2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\right]^8$ **44.** $\left[5\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)\right]^4$

In Exercises 45–56, (a) use DeMoivre's Theorem to find the indicated roots, (b) represent each of the roots graphically, and (c) express each of the roots in standard form.

45. Square roots: $16\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

46. Square roots: $9\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

47. Fourth roots: $16\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

48. Fifth roots: $32\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

49. Square roots: $-25i$

50. Fourth roots: $625i$

51. Cube roots: $-\frac{125}{2}(1 + \sqrt{3}i)$

52. Cube roots: $-4\sqrt{2}(1 - i)$

53. Cube roots: 8

54. Fourth roots: i

55. Fourth roots: 1

56. Cube roots: 1000

In Exercises 57–62, find all the solutions to the given equation and represent your solutions graphically.

57. $x^4 - i = 0$

58. $x^3 + 1 = 0$

59. $x^5 + 243 = 0$

60. $x^4 - 81 = 0$

61. $x^3 + 64i = 0$

62. $x^4 + i = 0$

63. Given two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ with $z_2 \neq 0$ prove that

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)].$$

64. Show that the complex conjugate of $z = r(\cos\theta + i\sin\theta)$ is

$$\bar{z} = r[\cos(-\theta) + i\sin(-\theta)].$$

65. Use the polar form of z and $\bar{z}$ in Exercise 64 to find the following.

(a) $z\bar{z}$

(b) $z/\bar{z},\ \bar{z} \neq 0$

66. Show that the negative of $z = r(\cos\theta + i\sin\theta)$ is

$$-z = r[\cos(\theta + \pi) + i\sin(\theta + \pi)].$$

67. (a) Let $z = r(\cos\theta + i\sin\theta) = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$. Sketch z, iz, and z/i in the complex plane.

(b) What is the geometric effect of multiplying a complex number z by i? What is the geometric effect of dividing z by i?

68. (Calculus) Recall that the Maclaurin series for e^x, $\sin x$, and $\cos x$ are

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

(a) Substitute $x = i\theta$ into the series for e^x and show that $e^{i\theta} = \cos\theta + i\sin\theta$.

(b) Show that any complex number $z = a + bi$ can be expressed in polar form as $z = re^{i\theta}$.

(c) Prove that if $z = re^{i\theta}$, then $\bar{z} = re^{-i\theta}$.

(d) Prove the amazing formula $e^{i\pi} = -1$.