Chapter 5

Applications of the definite integral to calculating volume and length

5.1

Find the volume of a cone whose height h is equal to its base radius r, by using the disc method. We will place the cone on its side, as shown in the Figure 5.1, and let x represent position along its axis.

(a) Using the diagram shown below (Figure 5.1), explain what kind of a curve in the xy plane we would use to *generate* the surface of the cone as a surface of revolution.

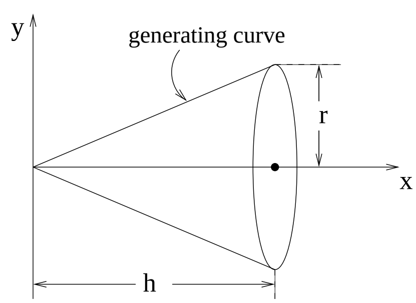

Figure 5.1: For problem 5.1

(b) Using the proportions given in the problem, specify the exact function $y = f(x)$ that we need to describe this "curve".

(c) Now find the volume enclosed by this surface of revolution for $0 \leq x \leq 1$.

(d) Show that, in this particular case, we would have gotten the same geometric object, and also the same enclosed volume, if we had rotated the "curve" about the y axis.

5.2

Find the volume of the cone generated by revolving the curve $y = f(x) = 1 - x$ (for $0 < x < 1$) about the y axis. Use the disk method, with disks stacked up along the y axis.

5.3

Find the volume of the "bowl" obtained by rotating the curve $y = 4x^2$ about the y axis for $0 \leq x \leq 1$.

5.4

On his wedding day, Kepler wanted to calculate the amount of wine contained inside a wine barrel whose shape is shown below in Figure 5.2. Use the disk method to compute this volume. You may assume that the function that generates the shape of the barrel (as a surface of revolution) is $y = f(x) = R - px^2$, for $-1 < x < 1$ where R is the radius of the widest part of the barrel. (R and p are both positive constants.)

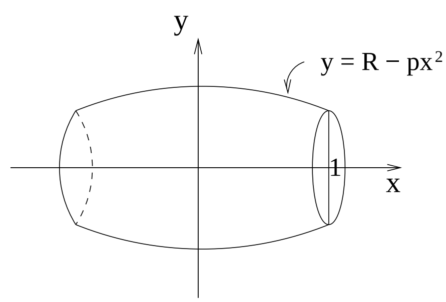

Figure 5.2: For problem 5.4

5.5

Consider the curve

$$y = f(x) = 1 - x^2 \quad 0 < x < 1$$

rotated about the y axis. Recall that this will form a shape called a paraboloid. Use the cylindrical shell method to calculate the volume of this shape.

5.6

Consider the curve $y = f(x) = \sqrt{1 - x^2}$ (which is a part of a circle of radius 1) over the interval $0 < x < 1$. Suppose this curve is rotated about the y axis to generate the top half of a sphere. Set up an integral which computes the volume of this hemisphere using the shell method, and compute the volume. (You will need to make a substitution to simplify the integral. To do this question, some techniques of anti-differentiation are required.)

5.7

Find the volume of the solid obtained by rotating the region bounded by the given curves $f(x)$ and $g(x)$ about the specified line.

(a) $f(x) = \sqrt{x-1}$, $g(x) = 0$, from $x = 2$ to $x = 5$, about the x-axis.

(b) $f(x) = \sqrt{x}$, $g(x) = x/2$, about the y-axis.

(c) $f(x) = 1/x$, $g(x) = x^3$, from $x = 1/10$ to $x = 1$, about the x-axis.

5.8

Let R= the region contained between $y = \sin(x)$ and $y = \cos(x)$ for $0 \le x \le \pi/2$. Write down the expression for the volume obtained by rotating R about

(a) x-axis ; and

(b) the line y=-1.

5.9

Suppose a lake has a depth of 40 meters at its deepest point and is bowl-shaped, with the surface of the bowl generated by rotating the curve $z = x^2/10$ around the z-axis. Here z is the height in meters above the lowest point of the bowl. The distribution of sediment in the lake is stratified by height along the water column. In other words, the density of sediment (in mass per unit volume) is a function of the form $s(z) = C(40 - z)$, where z is again vertical height in meters from the point at the bottom of the lake. Find the total mass of sediment in the lake (Your answer will have the constant C in it.). The volume of the lake is the volume above the curve $z = x^2/10$ and below $z = 40$.

5.10

Find the volume of a solid torus (donut shaped region) with radii r and R as shown in Figure 5.3. (Hint: There are several ways to do this. You can consider this as a surface of revolution and slice it up into little disks with holes ("washers") as shown.)

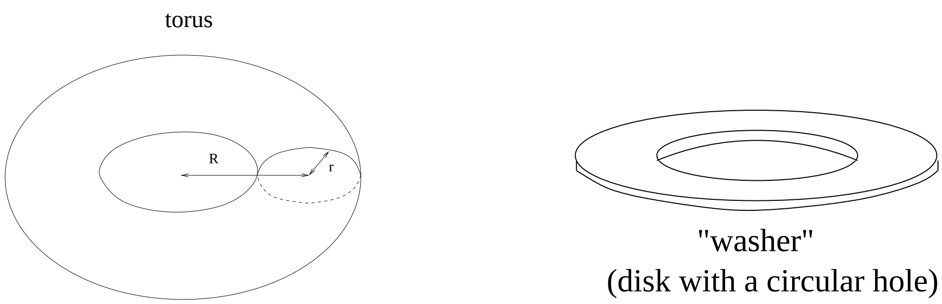

Figure 5.3: For problem 5.10

5.11

In this problem you are asked to find the volume of a height h pyramid with a square base of width w. (This is related to the Cheops pyramid problem, but we will use calculus.) Let the variable z stand for distance **down** the axis of the pyramid with $z = 0$ at the top, and consider "slicing" the pyramid along this axis (into horizontal slices). This will produce square "slices" (having area $A(z)$ and some width Δz). Calculate the volume of the pyramid as an integral by figuring out how $A(z)$ depends on z and integrating this function.

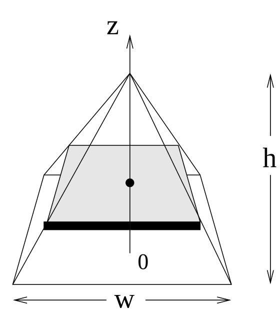

Figure 5.4: For problem 5.11

5.12

Set up the integral that represents the length of the following curves: Do not attempt to calculate the integral in any of these cases

(a)
$$y = f(x) = \sin(x) \quad 0 < x < 2\pi.$$

(b)
$$y = f(x) = \sqrt{x} \quad 0 < x < 1.$$

(c)
$$y = f(x) = x^n \quad -1 < x < 1.$$

5.13

Compute the length of the line $y = 2x + 1$ for $-1 < x < 1$ using the arc-length formula. Check your work by using the simple distance formula (or Pythagorean theorem).

5.14

Find the length of the curve

$$y = f(x) = x^{3/2} \qquad 0 < x < 1$$

(To do this question, some techniques of anti-differentiation are required.)

5.15

To do this question, some techniques of anti-differentiation are required.

(a) Use the chain rule to show that $F'(x) = \sec(x)$ is the derivative of the function

$$F(x) = \ln(\sec(x) + \tan(x))$$

So, what is the anti-derivative of $f(x) = \sec(x)$?

(b) Use integration by parts to show that

$$\int \sec^3(x)dx = \frac{1}{2}(\sec(x)\tan(x) + \ln(\sec(x) + \tan(x)) + C.$$

You will want to recall the trigonometric identity

$$1 + \tan^2(x) = \sec^2(x).$$

(c) Set up an integral to compute the arclength of the curve $y = x^2/2$ for $0 < x < 1$.

(d) Use the substitution $x = \tan(u)$ to reduce this arclength integral to an integral of the form

$$\int \sec^3(u)\ du.$$

5.16

Second Spreadsheet assignment: You are told that the **derivative** of a certain function is

$$f'(x) = 1 - 2\sin^2(x/3)$$

and that $f(0) = 0$. Use the spreadsheet to create one plot that contains all of the following graphs: (1) The graph of the function $y = f(x)$ (whose derivative is given to you). This should be plotted over the interval from 0 to 9. (2) The graph of the "element of arclength" $dl = \sqrt{1 + (f'(x))^2}dx$ showing how this varies across the same interval. (3) The graph of the cumulative length of the curve $L(x)$. Briefly indicate what you did to find the function in part (1). (You might consider how the spreadsheet would help you calculate the values of the desired function, $y = f(x)$, rather than trying to find an expression for it.)

5.17 work 1

A spring has a natural length of 16 cm. When it is stretched x cm beyond that, Hooke's Law states that the spring pulls back with a restoring force $F = kx$ dyne, where the constant k is called the spring constant, and represents the stiffness of the spring. For the given spring, 8 dyne of force are required to hold it stretched by 2 cm. How much work (dyne-cm) is done in stretching this spring from its natural length to a length 24 cm?

5.18 work 2

Calculate the work done in pumping water out of a parabolic container. Assume that the container is a surface of revolution generated by rotating the curve $y = x^2$ about the y axis, that the height of the water in the container is 10 units, that the density of water is $1gm/cm^3$ and that the force due to gravity is $F = mg$ where m is mass and $g = 9.8m/s^2$.

5.19 Energy

In a harmonic oscillator (such as a spring) the energy is partly kinetic (associated with motion), and partly potential (stored in the stretch of the spring). Energy conservation implies that the sum of the two forms (E) is constant so that

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E$$

where m is the mass on the spring, k the spring constant. (To do this question, some techniques of anti-differentiation are required.)

(a) Use the fact that the velocity is $v = dx/dt$ to show that this energy conservation equation implies that

$$\frac{dx}{dt} = \left(\frac{2}{m}E - \frac{k}{m}x^2\right)^{1/2}.$$

(b) Rewrite this in integral form by dividing both sides of the equation by the term in the square-root to obtain

$$\int \frac{dx}{\left(\frac{2}{m}E - \frac{k}{m}x^2\right)^{1/2}} = \int dt.$$

(c) Integrate both sides. You will need to use substitution, and the result will be an inverse trig function for x.

(d) Find the displacement x as a function of time t. Show that the motion of the spring is periodic.

5.20 Shear

A vertical shear is a transformation of the plane which takes the vertical line $x = a$ and pushes it upward by a distance sa (the constant s is called the strength of the shear). For instance, the rectangle shown below in Figure 5.5 is transformed into a trapezoid.

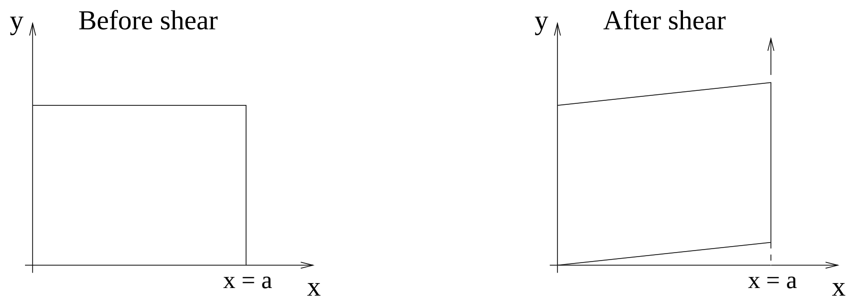

Figure 5.5: For problem 5.20

(a) Set up a definite integral which shows that the area of the trapezoid is the same as the area of the original rectangle. We say that the shear has preserved the area of the rectangle.

(b) Now consider a more general shape:

Figure 5.6: For problem 5.20(b)

Show that the area before the shear is the same as the area after the shear.

5.21 Surface Area

Calculate the surface area of a cone shaped surface obtained by rotating the curve $y = \sqrt{x}$ on the interval $[0, 2]$ around the x-axis. To do this question, some techniques of anti-differencation are required.