APPENDIX F

Complex Numbers

Operations with Complex Numbers

Some equations have no real solutions. For instance, the quadratic equation

$$x^2 + 1 = 0 \qquad \text{Equation with no real solution}$$

has no real solution because there is no real number x that can be squared to produce -1. To overcome this deficiency, mathematicians created an expanded system of numbers using the **imaginary unit** i, defined as

$$i = \sqrt{-1} \qquad \text{Imaginary unit}$$

where $i^2 = -1$. By adding real numbers to real multiples of this imaginary unit, we obtain the set of **complex numbers.** Each complex number can be written in the **standard form,** $a + bi$.

Definition of a Complex Number

For real numbers a and b, the number

$$a + bi$$

is a **complex number.** If $b \neq 0$, $a + bi$ is called an **imaginary number,** and bi is called a **pure imaginary number.**

To add (or subtract) two complex numbers, you add (or subtract) the real and imaginary parts of the numbers separately.

Addition and Subtraction of Complex Numbers

If $a + bi$ and $c + di$ are two complex numbers written in standard form, their sum and difference are defined as follows.

$$\textit{Sum: } (a + bi) + (c + di) = (a + c) + (b + d)i$$

$$\textit{Difference: } (a + bi) - (c + di) = (a - c) + (b - d)i$$

The **additive identity** in the complex number system is zero (the same as in the real number system). Furthermore, the **additive inverse** of the complex number $a + bi$ is

$$-(a + bi) = -a - bi \qquad \text{Additive inverse}$$

Thus, you have

$$(a + bi) + (-a - bi) = 0 + 0i = 0.$$

EXAMPLE 1 Adding and Subtracting Complex Numbers

a. $(3 - i) + (2 + 3i) = 3 - i + 2 + 3i \qquad$ Remove parentheses.

$$= 3 + 2 - i + 3i \qquad \text{Group like terms.}$$

$$= (3 + 2) + (-1 + 3)i$$

$$= 5 + 2i \qquad \text{Standard form}$$

NOTE Notice in Example 1b that the sum of two complex numbers can be a real number.

b. $2i + (-4 - 2i) = 2i - 4 - 2i \qquad$ Remove parentheses.

$$= -4 + 2i - 2i \qquad \text{Group like terms.}$$

$$= -4 \qquad \text{Standard form}$$

c. $3 - (-2 + 3i) + (-5 + i) = 3 + 2 - 3i - 5 + i$

$$= 3 + 2 - 5 - 3i + i$$

$$= 0 - 2i$$

$$= -2i$$

Many of the properties of real numbers are valid for complex numbers as well. Here are some examples.

> *Associative Properties of Addition and Multiplication*
> *Commutative Properties of Addition and Multiplication*
> *Distributive Property of Multiplication over Addition*

Notice below how these properties are used when two complex numbers are multiplied.

STUDY TIP Rather than trying to memorize the multiplication rule at the right, you can simply remember how the Distributive Property is used to multiply two complex numbers. The procedure is similar to multiplying two polynomials and combining like terms.

$$(a + bi)(c + di) = a(c + di) + bi(c + di) \qquad \text{Distributive}$$

$$= ac + (ad)i + (bc)i + (bd)i^2 \qquad \text{Distributive}$$

$$= ac + (ad)i + (bc)i + (bd)(-1) \qquad \text{Definition of } i$$

$$= ac - bd + (ad)i + (bc)i \qquad \text{Commutative}$$

$$= (ac - bd) + (ad + bc)i \qquad \text{Associative}$$

EXAMPLE 2   Multiplying Complex Numbers

a. $(3 + 2i)(3 - 2i) = 9 - 6i + 6i - 4i^2$ Product of binomials

$= 9 - 4(-1)$ $i^2 = -1$

$= 9 + 4$ Simplify.

$= 13$ Standard form

b. $(3 + 2i)^2 = 9 + 6i + 6i + 4i^2$ Product of binomials

$= 9 + 4(-1) + 12i$ $i^2 = -1$

$= 9 - 4 + 12i$ Simplify.

$= 5 + 12i$ Standard form

Notice in Example 2a that the product of two complex numbers can be a real number. This occurs with pairs of complex numbers of the form $a + bi$ and $a - bi$, called **complex conjugates.**

$$(a + bi)(a - bi) = a^2 - abi + abi - b^2i^2$$
$$= a^2 - b^2(-1)$$
$$= a^2 + b^2$$

To find the quotient of $a + bi$ and $c + di$ where c and d are not both zero, multiply the numerator and denominator by the conjugate of the denominator to obtain

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di}\left(\frac{c - di}{c - di}\right) = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}.$$

EXAMPLE 3   Dividing Complex Numbers

$$\frac{2 + 3i}{4 - 2i} = \frac{2 + 3i}{4 - 2i}\left(\frac{4 + 2i}{4 + 2i}\right)$$ Multiply by conjugate.

$$= \frac{8 + 4i + 12i + 6i^2}{16 - 4i^2}$$ Expand.

$$= \frac{8 - 6 + 16i}{16 + 4}$$ $i^2 = -1$

$$= \frac{1}{20}(2 + 16i)$$ Simplify.

$$= \frac{1}{10} + \frac{4}{5}i$$ Standard form

Complex Solutions of Quadratic Equations

When using the Quadratic Formula to solve a quadratic equation, you often obtain a result such as $\sqrt{-3}$, which you know is not a real number. By factoring out $i = \sqrt{-1}$, you can write this number in standard form.

$$\sqrt{-3} = \sqrt{3(-1)} = \sqrt{3}\sqrt{-1} = \sqrt{3}i$$

The number $\sqrt{3}i$ is called the principal square root of -3.

STUDY TIP The definition of principal square root uses the rule

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

for $a > 0$ and $b < 0$. The rule is not valid if *both* a and b are negative. For example,

$$\sqrt{-5}\sqrt{-5} = \sqrt{5}i\sqrt{5}i$$
$$= \sqrt{25}i^2$$
$$= 5i^2 = -5$$

whereas

$$\sqrt{(-5)(-5)} = \sqrt{25} = 5.$$

To avoid problems with multiplying square roots of negative numbers, be sure to convert to standard form *before* multiplying.

Principal Square Root of a Negative Number

If a is a positive number, the **principal square root** of the negative number $-a$ is

$$\sqrt{-a} = \sqrt{a}i.$$

EXAMPLE 4 Writing Complex Numbers in Standard Form

a. $\sqrt{-3}\sqrt{-12} = \sqrt{3}i\sqrt{12}i = \sqrt{36}i^2 = 6(-1) = -6$

b. $\sqrt{-48} - \sqrt{-27} = \sqrt{48}i - \sqrt{27}i = 4\sqrt{3}i - 3\sqrt{3}i = \sqrt{3}i$

c. $\left(-1 + \sqrt{-3}\right)^2 = \left(-1 + \sqrt{3}i\right)^2$

$$= (-1)^2 - 2\sqrt{3}i + \left(\sqrt{3}\right)^2(i^2)$$
$$= 1 - 2\sqrt{3}i + 3(-1)$$
$$= -2 - 2\sqrt{3}i$$

EXAMPLE 5 Complex Solutions of a Quadratic Equation

Solve $3x^2 - 2x + 5 = 0$.

Solution

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(5)}}{2(3)} \qquad \text{Quadratic Formula}$$

$$= \frac{2 \pm \sqrt{-56}}{6} \qquad \text{Simplify.}$$

$$= \frac{2 \pm 2\sqrt{14}i}{6} \qquad \text{Write in } i\text{-form.}$$

$$= \frac{1}{3} \pm \frac{\sqrt{14}}{3}i \qquad \text{Standard form}$$

Imaginary
axis

Figure F.1

Polar Form of a Complex Number

Just as real numbers can be represented by points on the real number line, you can represent a complex number

$$z = a + bi$$

at the point (a, b) in a coordinate plane (the **complex plane**). The horizontal axis is called the **real axis** and the vertical axis is called the **imaginary axis,** as shown in Figure F.1.

The **absolute value** of the complex number $a + bi$ is defined as the distance between the origin $(0, 0)$ and the point (a, b).

> ### The Absolute Value of a Complex Number
>
> The **absolute value** of the complex number $z = a + bi$ is given by
>
> $$|a + bi| = \sqrt{a^2 + b^2}.$$
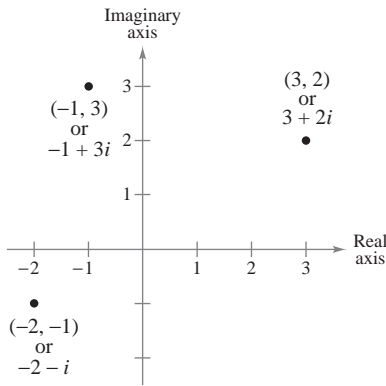
NOTE   If the complex number $a + bi$ is a real number (that is, if $b = 0$), then this definition agrees with that given for the absolute value of a real number.
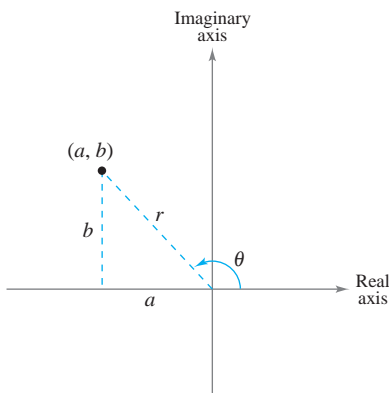
$$|a + 0i| = \sqrt{a^2 + 0^2} = |a|.$$

To work effectively with *powers* and *roots* of complex numbers, it is helpful to write complex numbers in **polar form.** In Figure F.2, consider the nonzero complex number $a + bi$. By letting θ be the angle from the positive x-axis (measured counterclockwise) to the line segment connecting the origin and the point (a, b), you can write

$$a = r \cos \theta \qquad \text{and} \qquad b = r \sin \theta$$

where $r = \sqrt{a^2 + b^2}$. Consequently, you have

$$a + bi = (r \cos \theta) + (r \sin \theta)i$$

from which you can obtain the **polar form of a complex number.**

Figure F.2

> ### Polar Form of a Complex Number
>
> The **polar form** of the complex number $z = a + bi$ is
>
> $$z = r(\cos \theta + i \sin \theta)$$
>
> where $a = r \cos \theta$, $b = r \sin \theta$, $r = \sqrt{a^2 + b^2}$, and $\tan \theta = b/a$. The number r is the **modulus** of z, and θ is called an **argument** of z.

NOTE   The polar form of a complex number is also called the **trigonometric form.** Because there are infinitely many choices for θ, the polar form of a complex number is not unique. Normally, θ is restricted to the interval $0 \le \theta < 2\pi$, although on occasion it is convenient to use $\theta < 0$.
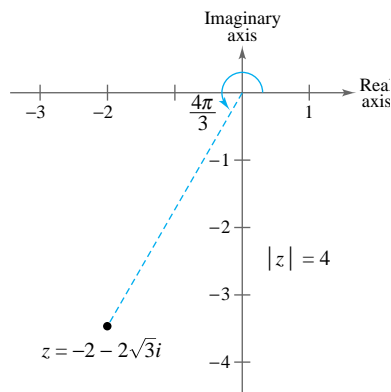
Imaginary
axis

$z = -2 - 2\sqrt{3}i$

$|z| = 4$

Figure F.3

EXAMPLE 6   Writing a Complex Number in Polar Form

Write the complex number $z = -2 - 2\sqrt{3}i$ in polar form.

Solution   The absolute value of z is

$$r = |-2 - 2\sqrt{3}i| = \sqrt{(-2)^2 + (-2\sqrt{3})^2} = \sqrt{16} = 4$$

and the angle θ is given by

$$\tan \theta = \frac{b}{a} = \frac{-2\sqrt{3}}{-2} = \sqrt{3}.$$

Because $\tan(\pi/3) = \sqrt{3}$ and because $z = -2 - 2\sqrt{3}i$ lies in Quadrant III, you choose θ to be $\theta = \pi + \pi/3 = 4\pi/3$. Thus, the polar form is

$$z = r(\cos \theta + i \sin \theta) = 4\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right).$$

(See Figure F.3.)

The polar form adapts nicely to multiplication and division of complex numbers. Suppose you are given two complex numbers

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad z_2 = r_2(\cos \theta_2 + i \sin \theta_2).$$

The product of z_1 and z_2 is

$$z_1 z_2 = r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)$$
$$= r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)].$$

Using the sum and difference formulas for cosine and sine, you can rewrite this equation as

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)].$$

This establishes the first part of the following rule. Try to establish the second part on your own.

Product and Quotient of Two Complex Numbers

Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ be complex numbers.

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \qquad \text{Product}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)], \quad z_2 \neq 0 \qquad \text{Quotient}$$

Note that this rule says that to multiply two complex numbers you multiply moduli and add arguments, whereas to divide two complex numbers you divide moduli and subtract arguments.

EXAMPLE 7   Multiplying Complex Numbers in Polar Form

Find the product of the complex numbers.

$$z_1 = 2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right) \qquad z_2 = 8\left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right)$$

Solution

$$z_1 z_2 = 2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right) \cdot 8\left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right)$$

$$= 16\left[\cos\left(\frac{2\pi}{3} + \frac{11\pi}{6}\right) + i \sin\left(\frac{2\pi}{3} + \frac{11\pi}{6}\right)\right]$$

$$= 16\left[\cos \frac{5\pi}{2} + i \sin \frac{5\pi}{2}\right]$$

$$= 16\left[\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right]$$

$$= 16[0 + i(1)] = 16i$$

Check this result by first converting to the standard forms $z_1 = -1 + \sqrt{3}i$ and $z_2 = 4\sqrt{3} - 4i$ and then multiplying algebraically.

EXAMPLE 8   Dividing Complex Numbers in Polar Form

Find the quotient z_1/z_2 of the complex numbers.

$$z_1 = 24(\cos 300° + i \sin 300°) \qquad z_2 = 8(\cos 75° + i \sin 75°)$$

Solution

$$\frac{z_1}{z_2} = \frac{24(\cos 300° + i \sin 300°)}{8(\cos 75° + i \sin 75°)}$$

$$= \frac{24}{8}\left[\cos(300° - 75°) + i \sin(300° - 75°)\right]$$

$$= 3\left[\cos 225° + i \sin 225°\right]$$

$$= 3\left[\left(-\frac{\sqrt{2}}{2}\right) + i\left(-\frac{\sqrt{2}}{2}\right)\right]$$

$$= -\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i$$

Powers and Roots of Complex Numbers

To raise a complex number to a power, consider repeated use of the multiplication rule.

$$z = r(\cos \theta + i \sin \theta)$$

$$z^2 = r^2(\cos 2\theta + i \sin 2\theta)$$

$$z^3 = r^3(\cos 3\theta + i \sin 3\theta)$$

$$\vdots$$

This pattern leads to the following important theorem, which is named after the French mathematician Abraham DeMoivre (1667–1754).

THEOREM A.4   DeMoivre's Theorem

If $z = r(\cos \theta + i \sin \theta)$ is a complex number and n is a positive integer, then

$$z^n = [r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta).$$

EXAMPLE 9   Finding Powers of a Complex Number

Use DeMoivre's Theorem to find $\left(-1 + \sqrt{3}i\right)^{12}$.

Solution   First convert to polar form.

$$-1 + \sqrt{3}i = 2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)$$

Then, by DeMoivre's Theorem, you have

$$\left(-1 + \sqrt{3}i\right)^{12} = \left[2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)\right]^{12}$$

$$= 2^{12}\left[\cos(12)\frac{2\pi}{3} + i \sin(12)\frac{2\pi}{3}\right]$$

$$= 4096(\cos 8\pi + i \sin 8\pi)$$

$$= 4096.$$

NOTE Notice in Example 9 that the answer is a real number.

Recall that a consequence of the Fundamental Theorem of Algebra is that a polynomial equation of degree n has n solutions in the complex number system. Each solution is an nth root of the equation. The **nth root** of a complex number is defined as follows.

Definition of nth Root of a Complex Number

The complex number $u = a + bi$ is an nth root of the complex number z if

$$z = u^n = (a + bi)^n.$$

STUDY TIP   The nth roots of a complex number are useful for solving some polynomial equations. For instance, you can use DeMoivre's Theorem to solve the polynomial equation

$$x^4 + 16 = 0$$

by writing -16 as $16(\cos \pi + i \sin \pi)$.

To find a formula for an nth root of a complex number, let u be an nth root of z, where

$$u = s(\cos \beta + i \sin \beta) \qquad \text{and} \qquad z = r(\cos \theta + i \sin \theta).$$

By DeMoivre's Theorem and the fact that $u^n = z$, you have

$$s_n(\cos n\beta + i \sin n\beta) = r(\cos \theta + i \sin \theta).$$

Taking the absolute values of both sides of this equation, it follows that $s^n = r$. Substituting back into the previous equation and dividing by r, you get

$$\cos n\beta + i \sin n\beta = \cos \theta + i \sin \theta.$$

Thus, it follows that

$$\cos n\beta = \cos \theta \qquad \text{and} \qquad \sin n\beta = \sin \theta.$$

Because both sine and cosine have a period of 2π, these last two equations have solutions if and only if the angles differ by a multiple of 2π. Consequently, there must exist an integer k such that

NOTE   When k exceeds $n - 1$, the roots begin to repeat. For instance, if $k = n$, the angle

$$\frac{\theta + 2\pi n}{n} = \frac{\theta}{n} + 2\pi$$

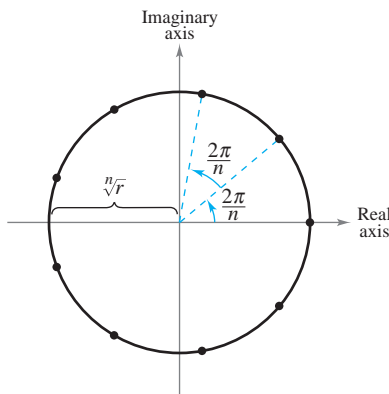is coterminal with θ/n, which is also obtained when $k = 0$.

$$n\beta = \theta + 2\pi k$$
$$\beta = \frac{\theta + 2\pi k}{n}.$$

By substituting this value for β into the polar form of u, you get the following result.

Figure F.4

THEOREM A.5 nth Roots of a Complex Number

For a positive integer n, the complex number $z = r(\cos\theta + i \sin\theta)$ has exactly n distinct nth roots given by

$$\sqrt[n]{r}\left(\cos \frac{\theta + 2\pi k}{n} + i \sin \frac{\theta + 2\pi k}{n}\right)$$

where $k = 0, 1, 2, \ldots, n - 1$.

This formula for the nth roots of a complex number z has a nice geometrical interpretation, as shown in Figure F.4. Note that because the nth roots of z all have the same magnitude $\sqrt[n]{r}$, they all lie on a circle of radius $\sqrt[n]{r}$ with center at the origin. Furthermore, because successive nth roots have arguments that differ by $2\pi/n$, the n roots are equally spaced along the circle.

EXAMPLE 10   **Finding the *n*th Roots of a Complex Number**

Find the three cube roots of $z = -2 + 2i$.

Solution

Because z lies in Quadrant II, the polar form for z is

$$z = -2 + 2i = \sqrt{8}\,(\cos 135° + i \sin 135°).$$

By the formula for *n*th roots, the cube roots have the form

$$\sqrt[6]{8}\left(\cos \frac{135° + 360°k}{3} + i \sin \frac{135° + 360°k}{3}\right).$$

Finally, for $k = 0$, 1, and 2, you obtain the roots

$$\sqrt{2}\,(\cos 45° + i \sin 45°) = 1 + i$$
$$\sqrt{2}\,(\cos 165° + i \sin 165°) \approx -1.3660 + 0.3660i$$
$$\sqrt{2}\,(\cos 285° + i \sin 285°) \approx 0.3660 - 1.3660i.$$

EXERCISES FOR APPENDIX F

In Exercises 1–24, perform the operation and write the result in standard form.

1. $(5 + i) + (6 - 2i)$

2. $(13 - 2i) + (-5 + 6i)$

3. $(8 - i) - (4 - i)$

4. $(3 + 2i) - (6 + 13i)$

5. $\left(-2 + \sqrt{-8}\right) + \left(5 - \sqrt{-50}\right)$

6. $\left(8 + \sqrt{-18}\right) - \left(4 + 3\sqrt{2}i\right)$

7. $13i - (14 - 7i)$

8. $22 + (-5 + 8i) + 10i$

9. $-\left(\frac{3}{2} + \frac{5}{2}i\right) + \left(\frac{5}{3} + \frac{11}{3}i\right)$

10. $(1.6 + 3.2i) + (-5.8 + 4.3i)$

11. $\sqrt{-6} \cdot \sqrt{-2}$

12. $\sqrt{-5} \cdot \sqrt{-10}$

13. $\left(\sqrt{-10}\right)^2$

14. $\left(\sqrt{-75}\right)^2$

15. $(1 + i)(3 - 2i)$

16. $(6 - 2i)(2 - 3i)$

17. $6i(5 - 2i)$

18. $-8i(9 + 4i)$

19. $\left(\sqrt{14} + \sqrt{10}i\right)\left(\sqrt{14} - \sqrt{10}i\right)$

20. $\left(3 + \sqrt{-5}\right)\left(7 - \sqrt{-10}\right)$

21. $(4 + 5i)^2$

22. $(2 - 3i)^2$

23. $(2 + 3i)^2 + (2 - 3i)^2$

24. $(1 - 2i)^2 - (1 + 2i)^2$

In Exercises 25–32, write the conjugate of the complex number. Multiply the number and its conjugate.

25. $5 + 3i$

26. $9 - 12i$

27. $-2 - \sqrt{5}i$

28. $-4 + \sqrt{2}i$

29. $20i$

30. $\sqrt{-15}$

31. $\sqrt{8}$

32. $1 + \sqrt{8}$

In Exercises 33–46, perform the operation and write the result in standard form.

33. $\dfrac{6}{i}$

34. $-\dfrac{10}{2i}$

35. $\dfrac{4}{4 - 5i}$

36. $\dfrac{3}{1 - i}$

37. $\dfrac{2 + i}{2 - i}$

38. $\dfrac{8 - 7i}{1 - 2i}$

39. $\dfrac{6 - 7i}{i}$

40. $\dfrac{8 + 20i}{2i}$

41. $\dfrac{1}{(4 - 5i)^2}$

42. $\dfrac{(2 - 3i)(5i)}{2 + 3i}$

43. $\dfrac{2}{1 + i} - \dfrac{3}{1 - i}$

44. $\dfrac{2i}{2 + i} + \dfrac{5}{2 - i}$

45. $\dfrac{i}{3 - 2i} + \dfrac{2i}{3 + 8i}$

46. $\dfrac{1 + i}{i} - \dfrac{3}{4 - i}$

In Exercises 47–54, use the Quadratic Formula to solve the quadratic equation.

47. $x^2 - 2x + 2 = 0$

48. $x^2 + 6x + 10 = 0$

49. $4x^2 + 16x + 17 = 0$

50. $9x^2 - 6x + 37 = 0$

51. $4x^2 + 16x + 15 = 0$

52. $9x^2 - 6x - 35 = 0$

53. $16t^2 - 4t + 3 = 0$

54. $5s^2 + 6s + 3 = 0$

In Exercises 55–62, simplify the complex number and write it in standard form.

55. $-6i^3 + i^2$

56. $4i^2 - 2i^3$

57. $-5i^5$

58. $(-i)^3$

59. $\left(\sqrt{-75}\right)^3$

60. $\left(\sqrt{-2}\right)^6$

61. $\dfrac{1}{i^3}$

62. $\dfrac{1}{(2i)^3}$

In Exercises 63–68, plot the complex number and find its absolute value.

63. $-5i$

64. -5

65. $-4 + 4i$

66. $5 - 12i$

67. $6 - 7i$

68. $-8 + 3i$

In Exercises 69–76, represent the complex number graphically, and find the polar form of the number.

69. $3 - 3i$

70. $2 + 2i$

71. $\sqrt{3} + i$

72. $-1 + \sqrt{3}i$

73. $-2\left(1 + \sqrt{3}i\right)$

74. $\frac{5}{2}\left(\sqrt{3} - i\right)$

75. $6i$

76. 4

In Exercises 77–82, represent the complex number graphically, and find the standard form of the number.

77. $2(\cos 150° + i \sin 150°)$

78. $5(\cos 135° + i \sin 135°)$

79. $\frac{3}{2}(\cos 300° + i \sin 300°)$

80. $\frac{3}{4}(\cos 315° + i \sin 315°)$

81. $3.75\left(\cos \dfrac{3\pi}{4} + i \sin \dfrac{3\pi}{4}\right)$

82. $8\left(\cos \dfrac{\pi}{12} + i \sin \dfrac{\pi}{12}\right)$

In Exercises 83–86, perform the operation and leave the result in polar form.

83. $\left[3\left(\cos \dfrac{\pi}{3} + i \sin \dfrac{\pi}{3}\right)\right]\left[4\left(\cos \dfrac{\pi}{6} + i \sin \dfrac{\pi}{6}\right)\right]$

84. $\left[\dfrac{3}{2}\left(\cos \dfrac{\pi}{2} + i \sin \dfrac{\pi}{2}\right)\right]\left[6\left(\cos \dfrac{\pi}{4} + i \sin \dfrac{\pi}{4}\right)\right]$

85. $\left[\frac{5}{3}(\cos 140° + i \sin 140°)\right]\left[\frac{2}{3}(\cos 60° + i \sin 60°)\right]$

86. $\dfrac{\cos(5\pi/3) + i \sin(5\pi/3)}{\cos \pi + i \sin \pi}$

In Exercises 87–94, use DeMoivre's Theorem to find the indicated power of the complex number. Express the result in standard form.

87. $(1 + i)^5$

88. $(2 + 2i)^6$

89. $(-1 + i)^{10}$

90. $(1 - i)^{12}$

91. $2\left(\sqrt{3} + i\right)^7$

92. $4\left(1 - \sqrt{3}i\right)^3$

93. $\left(\cos \dfrac{5\pi}{4} + i \sin \dfrac{5\pi}{4}\right)^{10}$

94. $\left[2\left(\cos \dfrac{\pi}{2} + i \sin \dfrac{\pi}{2}\right)\right]^8$

In Exercises 95–100, (a) use Theorem A.5 on page A64 to find the indicated roots of the complex number, (b) represent each of the roots graphically, and (c) express each of the roots in standard form.

95. Square roots of $5(\cos 120° + i \sin 120°)$

96. Square roots of $16(\cos 60° + i \sin 60°)$

97. Fourth roots of $16\left(\cos \dfrac{4\pi}{3} + i \sin \dfrac{4\pi}{3}\right)$

98. Fifth roots of $32\left(\cos \dfrac{5\pi}{6} + i \sin \dfrac{5\pi}{6}\right)$

99. Cube roots of $-\dfrac{125}{2}\left(1 + \sqrt{3}i\right)$

100. Cube roots of $-4\sqrt{2}(1 - i)$

In Exercises 101–108, use Theorem A.5 on page A64 to find all the solutions of the equation and represent the solutions graphically.

101. $x^4 - i = 0$

102. $x^3 + 1 = 0$

103. $x^5 + 243 = 0$

104. $x^4 - 81 = 0$

105. $x^3 + 64i = 0$

106. $x^6 - 64i = 0$

107. $x^3 - (1 - i) = 0$

108. $x^4 + (1 + i) = 0$