Euler and Differentials

Megan Levine

Abstract

This is a review of Anthony P. Ferzola's article "Euler and Differentials," which discusses Euler's contributions to mathematics and differentials in calculus.

Anthony P. Ferzola expressed his respect and awe for Leonard Euler in his article "Euler and Differentials," published by The College Mathematics Journal(March 1994). Ferzola discussed the history of Euler and the succession of his works. He commented on and explained the computations and concepts Euler was working with before calculus as we know it was established.

1 Differentials as Absolute Zeros

Euler did not like the vague and ambiguous explanations for the infinitesmal, like,

> smaller than any given quantity in mathematics.

To clarify his view about this he once stated

> To those who ask what the infinitely small quantity in mathematics is, we answer it is actually equal to zero.

Through the rationalizations

$$n \cdot = 0 \tag{1}$$

$$\frac{n}{1} = \frac{0}{0}, \tag{2}$$

the quotient $\frac{0}{0}$ can represent any arbitrary value. Euler used the Leibnizian notation to distinguish between the zero in the numerator and the zero in the denominator. He also rationalized that since $dx = 0$, $(dx)^2 = 0$, and since $(dx)^2$ is of higher order than one,

$$dx + (dx)^2 = dx \tag{3}$$

$$\frac{dx + (dx)^2}{dx} = 1 \tag{4}$$

$$1 + dx = 1, \quad \text{therefore} \tag{5}$$

$$dx + (dx)^{n+1} = dx, \text{ for all } n > 0 \tag{6}$$

Euler was able to dismiss higher order infinitesmals because they were basically equal to zero. This is a technique of Euler's shown throughout Ferzola's article.

2 Computing of Elementary Functions

In this section of the article, Ferzola walks the reader through Euler's derivations for the important calculus rules of differentiating. The power rule, product rule, quotient rule, and the derivatives for transcendentals and trigonometric functions are all covered by Ferzola. In each of the reasonings for these derivations, Euler goes through the same process of adding an infinitesmal to the variable he is differentiating with respect to. Then he subtracts the original expression from it. Euler takes advantage of infinite series expansions but almost all of the terms in the series end up omitted as higher order infinitesmals.

2.1 The Power Rule

if $y = x^n$, then

$$dy = (x + dx)^n - x^n \tag{7}$$

$$= nx^{n-1}dx + \frac{n(n-1)}{1}\, x^{n-2}(dx)^2 + \cdots \tag{8}$$

$$dy = nx^{n-1}dx \tag{9}$$

2.2 The Product Rule

if $y = pq$,

$$d(pq) = (p + dp)(q + dq) - pq \tag{10}$$

$$= pq + pdq + qdp + dpdq - pq, \tag{11}$$

$dpdq$ is omitted because it is a higher order infinitesmal

$$d(pq) = pdq + qdp$$

2.3 The Quotient Rule

First Euler found it necessary to define the denominator of the quotient when an infinitesmal is added to it.

$$\frac{1}{q + dq} = \frac{1}{q}\left(\frac{1}{1 + \frac{dq}{q}}\right) \tag{12}$$

$$= \frac{1}{q}(1 - \frac{dq}{q} + \frac{dq^2}{q^2} - \cdots) \tag{13}$$

$$= \frac{1}{q} - \frac{dq}{q^2} \tag{14}$$

Through the "unique use of geometric series," Euler came to the conclusion that

$$\left(\frac{1}{q+dq}\right)$$

could be rewritten as

$$\frac{1}{q}-\frac{dq}{q^2}$$

. Then it is possible for Euler to complete his differentiation process with a quotient.

$$d\left(\frac{p}{q}\right)=\frac{p+dp}{q+dq}-\frac{p}{q} \tag{15}$$

$$=(p+dp)\left(\frac{1}{q+dq}\right)-\frac{p}{q} \tag{16}$$

$$=(p+dp)(\frac{1}{q}-\frac{dq}{q^2})-\frac{p}{q} \tag{17}$$

$$=\frac{p+dp}{q}-\frac{pdq+dpdq}{q^2}-\frac{p}{q} \tag{18}$$

Combining terms by finding a common denominator, and omitting $dpdq$ as a higher order infinitesmal, can simplify this equation greatly.

$$d\left(\frac{p}{q}\right)=\frac{qp+qdp-pdq-pq}{q^2} \tag{19}$$

$$d(\frac{p}{q})=\frac{qdp-pdq}{q^2} \tag{20}$$

2.4 Transcendentals or Natural Logarithms

Euler first used Mercator's series to expand $\ln 1+z$.

$$\ln 1+z=z-\frac{z^2}{2}+\frac{z^3}{3}-\cdots$$

and using $y=\ln x$, then

$$dy=\ln(x+dx)-\ln x \tag{21}$$

$$=\ln(\frac{x+dx}{x}) \tag{22}$$

$$=\ln(1+\frac{dx}{x}) \tag{23}$$

$$dy=\frac{dx}{x}-\frac{dx^2}{x^2}+\frac{dx^3}{x^3}-\cdots \tag{24}$$

, and omitting higher order differentials

$$dy=\frac{dx}{x}$$

2.5 Trigonometric Functions

To solve for the derivatives of cosine and sine, Euler "explicitly used the sine and cosine series." Where

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{25}$$

and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Now all Euler had to do was substitute dx in for x and omitt the higher order infinitesmals to get the result

$$\sin(dx) = dx, \text{and} \tag{26}$$

$$\cos(dx) = 1, \tag{27}$$

which is logical if we remember dx is basically zero. Then Euler took advantage of the Trigonometric Sum Identity to add an infinitesmal to x.

$$dy = \sin(x + dx) - \sin x \tag{28}$$

$$= \sin x \ \cos dx + \cos x \ \sin dx - \sin x \tag{29}$$

$$= \sin x + \cos x dx - \sin x \tag{30}$$

$$dy = \cos x dx. \tag{31}$$

The results from Eq. (26) and Eq. (27) were substituted into Eq. (29) to simplify the derivative. The same process and technique is used to prove that

$$d(\cos x) = -\sin x dx$$

3 Integration

> In Euler's three volume *Institutiones calculi integralis* (1768–1770), he defined integration, like Leibniz and Johann Bernoulli, as the formal inverse of differentiation. [1]

4 The Total Differential

Ferzola also wrote about Euler's work on multi-variable differentials. Euler decided that any expression V of three variables x, y, z has the differential

$$dV = pdx + qdy + rdz,$$

where p, q, r are functions of the three variables x, y, z. He was able to generalize this statement for an arbitrary amount of variables.

Also, Ferzola explained Euler's work with multiple integrals. Euler saw these as "volumes" and also "interpreted $dxdy$ as an area element of $\mathbf{R}$," where $\mathbf{R}$ is the "bounded domain ... enclosed by arcs in the xy plane."

5 Conclusion

Ferzola comments throughout the paper that Euler's argumens are not logically sound, but "the mind behind it all is that of a unique master." I thought that Euler's technique of adding an infinitesmal to the variable or main parts of equation, and then subtracting the original expression made sense. It seems like a very logical way to look at differentiation. We are trying to find out how much the expression changes when the variable changes by an infinitely small quantity.

Ferzola explained Euler's style of textbook writing by saying that,

> Euler frequently let his readers in on his thought processes, even when the procedures seemed fruitless. This was mathematics being done for all to see, not a slick modern textbook treatment. There was no taking down the scaffolding á la Gauss. [1]

References

[1] Ferzola,Anthony *Euler and Differentials*, College Mathematics Journal