SIGNAL PROCESSING & SIMULATION NEWSLETTER

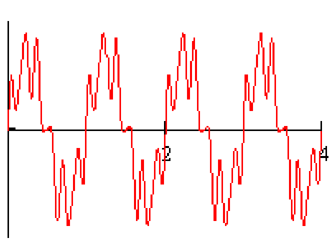
Fourier analysis made Easy *Part 1*

Jean Baptiste Joseph, Baron de Fourier, 1768 - 1830

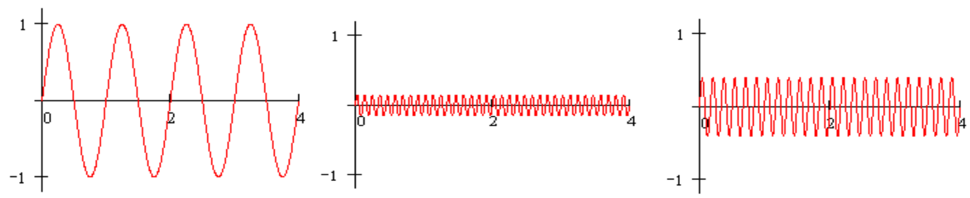
While studying heat conduction in materials, Baron Fourier (a title given to him by Napoleon) developed his now famous Fourier series, approximately 120 years after Newton published the first book on calculus. It took Fourier another twenty years to develop the Fourier transform which made the theory applicable to a variety of disciplines such as signal processing where Fourier analysis is now an essential tool. Fourier did little to develop the concept further and most of that work was done by Euler, LaGrange, Laplace and others. Fourier analysis is now also used in thermal analysis, image processing, quantum mechanics and physics.

Why do we need to do Fourier analysis – In communications, we can state the problem at hand this way; we send an information-laced signal over a medium. The medium and the hardware corrupt this signal. The receiver has to figure out from the received signal which part of the corrupted received signal is the information signal and which part the extraneous noise and distortion. The transmitted signals have well defined spectral content, so if the receiver can do a spectral analysis of the received signal then it can extract the information. This is what Fourier analysis allows us to do. Fourier analysis can look at an unknown signal and do an equivalent of a chemical analysis, identifying the various frequencies and their relative “quantities” in the signal.

Fourier noticed that you can create some really complicated looking waves by just summing up simple sine and cosine waves. For example, the wave in Figure 1a is sum of the just three sine waves shown in Figures 1b, 1c and 1d of assorted frequencies and amplitudes.



(a) - A complicated looking wave



(b) - Sine wave 1 **(c)- Sine wave 2** **(d) - Sine wave 3**

Figure 1 - Sine waves

Let's look at signal 1a in three dimensions. With time progressing to the right we see the amplitude going up and down erratically, we are looking at the signal in **Time domain.** From this angle, we see the sum of the three sine waves as shown in Fig (1b,c,d).

When we look at the same signal from the side along the z-axis, what we see are the three sine waves of different frequencies. We also see the amplitude but only as a line with its maximum excursion. This view of the signal from this point of view is called the **Frequency Domain**. Another name for it is the **Signal Spectrum**.

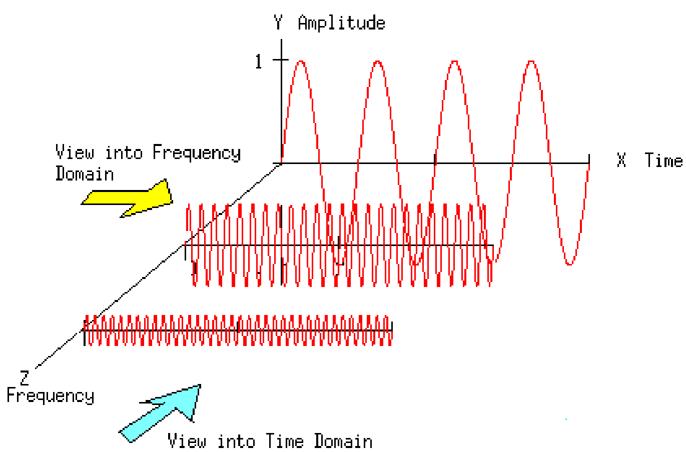


Figure 3 - Looking at signals from two different points of view

The concept of spectrum came about from the realization that any arbitrary wave is really a summation of many different frequencies. The spectrum of the composite wave f(t) of Fig (1) is composed of just three frequencies and can be drawn as in Fig (3.1).

This is called a one-sided **magnitude spectrum**. One-sided not because anything has been left out of it, but because only positive frequencies are represented. (*So what is a negative frequency? Is there such a thing? We will discuss this in more detail in later. For now, suffice it to say that a negative frequency is simply a frequency which is lagging in phase.*)

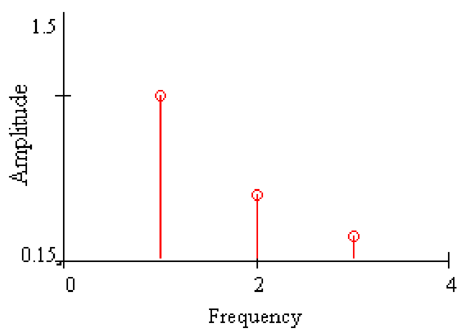


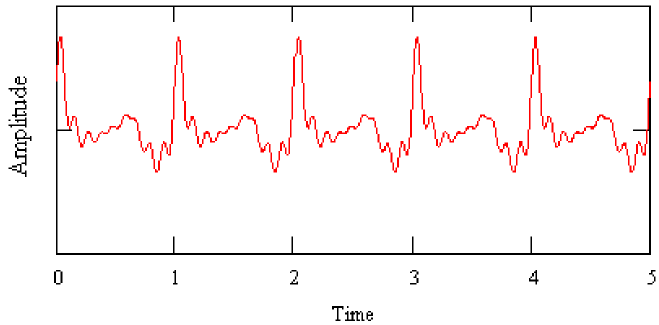
Figure 3.1 - The Frequency Domain spectrum of wave in Figure 1

Now let's look at the signal in frequency domain. Think of it as a recipe, with x-axis showing the ingredient and the y-axis, how much of that ingredients. The x-axis for a signal would show the different frequencies in the signal and y-axis the amplitude of each of those frequencies.

Let's expand on this concept. V-8 juice for example has many different ingredients such as celery juice, salt, water, spices, etc.. We can remove most of these ingredients one by one and the remaining liquid would still taste essentially like V-8. What we can not remove and have the item still retain its primary character is called the **fundamental component**. In V-8, that is tomato juice.

Signals carrying information, similarly, have a fundamental frequency along with other lesser important frequencies. A noisy signal on the other has no single fundamental frequency. It has a flat spectrum. All frequencies are present in the signal in the same quantities. So a spectrum does not necessarily have a fundamental component. The spectrum of such a signal would be flat.

Let's take the following complicated looking wave.



This wave is periodic with a period = 1 sec.

Figure 4 - Another really complicated looking wave

The first thing we notice is that the wave is periodic. Fourier analysis tells us that any arbitrary wave such as the above that is periodic, can be represented by a sum of other simpler waves.

Let's try summing a bunch of sine waves to see what they look like.

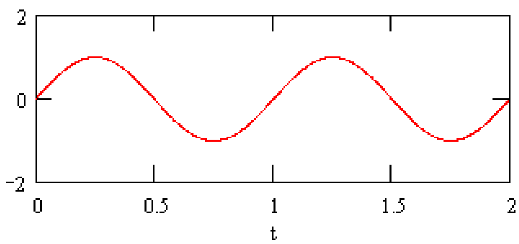


Figure 5a - This is a wave of frequency 1 Hz , amplitude = 1

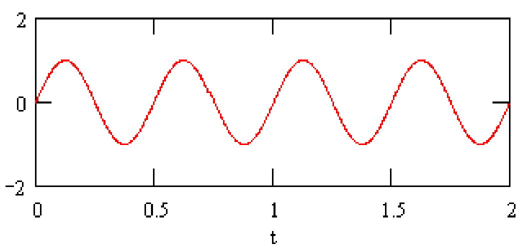


Figure 5b - This is a wave of frequency 2 Hz , amplitude = 1

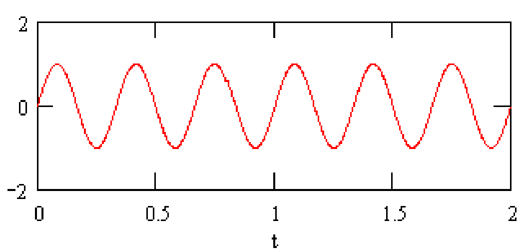


Figure 5c - This is a wave of frequency 3 Hz , amplitude = 1

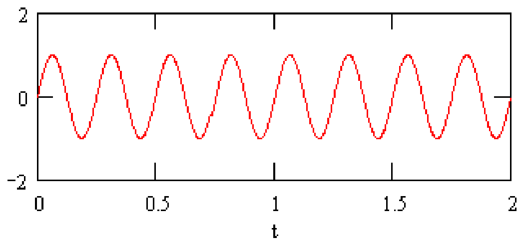


Figure 5d - This is a wave of frequency 4 Hz , amplitude = 1

Each of the waves here have frequencies that are integer multiples. In more scientific words, we say that they are harmonic to each other, similar to musical notes which are also called harmonic.

- ***What is a harmonic*** – It is a frequency that is integer multiple of the other frequency. Waves of frequency 2 and 4 Hz are harmonics to a wave of frequency 1 Hz since they are both its integer multiples. Frequencies 2.4 and 3.6 Hz are harmonics to a wave of frequency 1.2 Hz since they are both integer multiples of 1.2 Hz.
- When the multiple factor is even, the harmonic is called an **even harmonic** and when the factor is odd, it is called the **odd harmonic**.
- Frequencies 66, 110, 154 Hz are odd harmonics of frequency 22 Hz, whereas 44, 88 and 132 Hz are even harmonics.

We write the sum of N such harmonics as

$$f(t) = \sum_{n=1}^{N} \sin(n\omega t) \qquad (1)$$

Each wave has a frequency that is integer multiple of the starting frequency ω, which is equal to $2\pi(1)$ in this case since f = 1 Hz. Here is what a sum of four sine waves of equal amplitude, each starting with a phase of 0 degrees at time 0 looks like.

$$f(t) = \sin(1\omega t) + \sin(2\omega t) + \sin(3\omega t) + \sin(4\omega t)$$

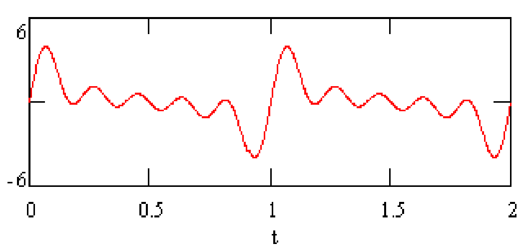


Figure 6 - This is the sum of all four of the above sine waves.

If we keep going and add a large number of sine waves of equal amplitude, the summation approaches an impulse function as shown below for N = 25. Since we added together 30 sine waves of amplitude 1, the maximum amplitude is 25.

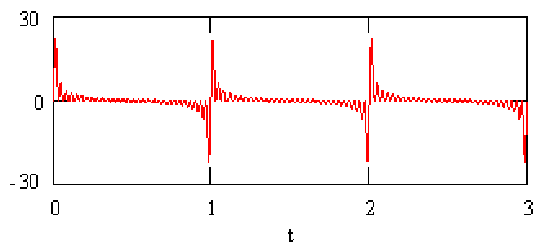


Figure 7 - This is the sum of 25 sine waves.

In the graph above, we allowed the amplitude of each harmonic to be one. Going to the next level of abstraction, it is obvious that to represent an arbitrary wave, we need to allow the amplitude of each component to vary. Otherwise, all we will get is the scaled version of the signal in Fig (7). So we modify equation (1) by introducing a coefficient a_n to represent the amplitude of the n_{th} sine wave as follows:

$$f(t) = \sum_{n=1}^{N} a_n \sin(n\,\omega t) \qquad (2)$$

The coefficient a_n allows us to vary the amplitude of each harmonic $f_n(t) = \sin(n\omega t)$ to create a variety of waves. Here is what one particular wave which is the sum of four sine waves of unequal amplitude looks like.

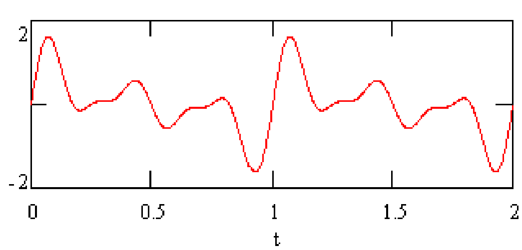


Figure 8 - Sum of four sine waves of unequal amplitude

But looking at the original wave, f(t) in Fig (4), we see that it starts at a non-zero value. No matter how many sine waves we add together, we can not replicate this wave because sine waves are always zero at time zero. But if we add some cosine waves to the sum in equation (2) which do not start at zero, we may be able to create the wave of Figure 2.

So let's add a bunch of cosine waves of varying amplitudes to our f(t) equation.

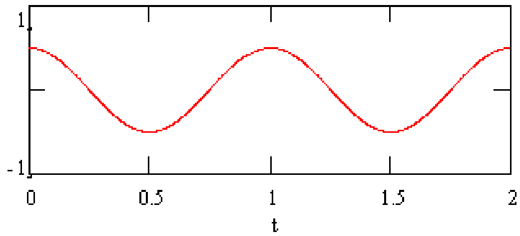


Figure 9a - A cosine wave of frequency 1 Hz, amplitude = 1

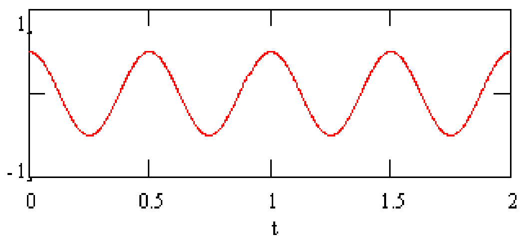


Figure 9b - A cosine wave of frequency 2 Hz, amplitude = 1

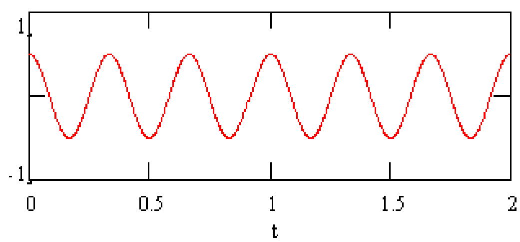


Figure 9c - A cosine wave of frequency 3 Hz, amplitude = 1

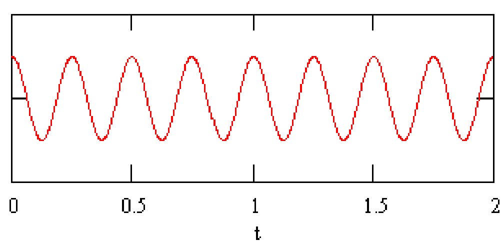


Figure 9d - A cosine wave of frequency 4 Hz, amplitude = 1

Once again the sum of the cosine waves of equal amplitude looks like this.

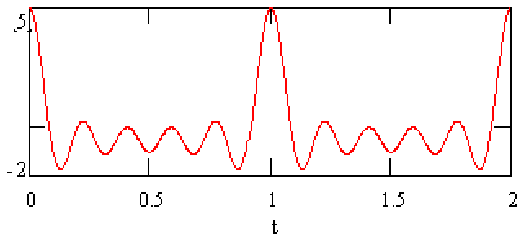


Figure 10 - Sum of four cosine waves of equal amplitude

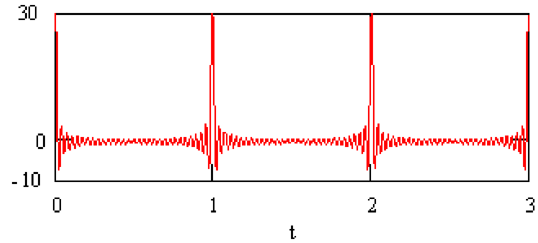


Figure 11 - Sum of 30 cosine waves of equal amplitude

A sum of 30 cosine waves looks like as in Fig (11). It approaches an impulse function just as the sum of sine waves did but this one is an even function.

- **Even function** – The function that is symmetrical about the y-axis. Cosine wave is an even function.
- **Odd function** – The function that is not symmetrical about the y-axis. Sine wave is an odd function.

The sum of the cosines is an even function. Contrast this with Fig (7), the sum of sines, which is an odd function. These characteristics, odd and even, are useful when looking at real and imaginary components of signals.

Now let's allow the amplitude of each cosine wave to vary. Here is what one particular sum of four cosines of unequal amplitudes looks like.

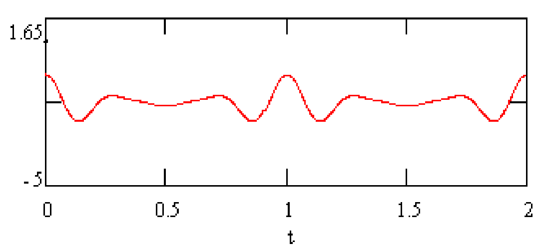


Figure 12 - Sum of four cosine waves of unequal amplitude.

Now let's modify equation (2) to add the cosine waves.

$$f(t)=\sum_{n=1}^{N} a_n \sin(n\,\omega t)+\sum_{n=1}^{N} b_n \cos(n\,\omega t) \qquad (3)$$

The coefficients b_n allow us to vary the amplitude of each cosine wave. Putting this equation to work, we see in the following figure the sum of four sine and four cosine waves.

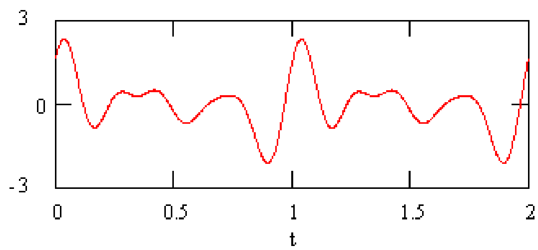


Figure 13 - Sum of five sine and cosine waves of unequal amplitudes

We are very close to completing our equation for arbitrary periodic waves. There is only one remaining issue. Sums of sine and cosines are always symmetrical about the x-axis so there is no possibility of representing a wave with a dc offset. To do that we add a constant, a_0 to the equation. This constant moves the whole wave up (or down) along the y-axis offset.

$$f(t)=a_0+\sum_{n=1}^{N} a_n \sin(n\omega t)+\sum_{n=1}^{N} b_n \cos(n\omega t) \qquad (4)$$

The coefficient a_0 provides us with the needed dc offset from zero. Now with this equation we can fully describe any periodic wave, no matter how complicated looking it is. All arbitrary but periodic waves are composed of just plain and ordinary sines and cosines and can de composed in its constituent frequencies..

Equation (4) is called the Fourier Series equation. The coefficients a_0, a_n, and b_n are called the Fourier Series Coefficients.

<u>An equation with many faces</u>

There are several different ways to write the Fourier series. One common representation is by linear frequency instead of the radial frequency. Replace ω by $2\pi f$ and then write the equation as

$$f(t)=a_0+\sum_{n=1}^{N} a_n \sin(2\pi nft)+\sum_{n=1}^{N} b_n \cos(2\pi nft) \qquad (5)$$

The Fourier series equation allows us to represent any wave, low or high frequency, baseband or passband, large bandwidth or very small. N is the number of harmonics used in the summation. This is a variable and we can choose it to be anything, but for complete representation, N is set to infinity. This makes the equation completely general and we can represent even noise signals this way. The harmonics themselves do not have to be of integer frequencies such 1, 2, 3 etc.. The starting frequency can be any real or imaginary number. However, the harmonics of the starting frequency ARE its integer multiples.

$$f_n(t) = n\,f(t) = n\frac{1}{T}$$

f(t) the smallest frequency is called the **resolution frequency**, determines how finely we decompose the signal. It can be any arbitrary number, say for example 2.35. From that point on, the next harmonic is 2 times this, next one 3 times and so on. T, is the period of the first wave we pick, and each f_n is an integer multiple of the inverse of that period. We can also start anywhere. We can pick a small resolution frequency and then start the analysis with the 100^{th} harmonic for example.

Replace f_n by n/T, where T is the period and replace N by ∞ to write equation (5) in a different from.

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \sin\left(2\pi t n/T\right) + b_n \cos\left(2\pi t n/T\right) \qquad (6)$$

We can also convert all sine waves and make them cosine waves by adding a half-period phase shift. The cosine representation, used often in signal processing is written by adding a phase term to the equation.

$$\sin(2\pi ft) = \cos(2\pi ft + \pi/2)$$

To create the f(t) we would add two cosine waves of the same frequency, except the one of them would have a $\pi/2$ phase shift (that's a sine wave, really.) Now we have only cosines. The name of the coefficient has been changed to c_n, to reduce confusion between this term and the terms a_n and b_n. a_0 and C_0 would be exactly the same as a_0.

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(2\pi f_n t + \phi_n)$$

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(w_n t + \phi_n) \qquad (6a)$$

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(\frac{2\pi n}{T} t + \phi_n)$$

In complex representation, the Fourier equation is written as

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\pi t/T} \qquad (7)$$

Complex notation, first given by Euler, is most useful-albeit scary-looking form. In next part, we look at how it is derived and used for signal processing.

All these different representations of the Fourier Series (4), (5), (6), (6a) and (7) are <u>identical</u> and mean exactly the same thing.

How to compute the Fourier Coefficients of an arbitrary wave

In signal processing, we are interested in spectral components of a signal. We want to know how many sines and cosines make up our signal and what their amplitudes are. Alternatively, what we really want are the Fourier coefficients of our signal. Once we know the Fourier coefficients, we know which frequencies are present in the signal and in what quantities. This is similar to doing chemical analysis on a compound, figuring out what elements are there and what relative quantity.

How do we compute the Fourier coefficients?

Computing a_0

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \sin \omega_n t + b_n \cos \omega_n t)$$

The constant a_0 in the Fourier equation above represents the dc offset. But before we compute it, let's take a look at one particular property of the sine and cosine waves.

Both sine and cosine wave are symmetrical about the x-axis. When you integrate a sine or a cosine wave over one period, you will always get zero. The areas above the x-axis cancels out the areas below it. This is always true over one period as we can see in the figure below.

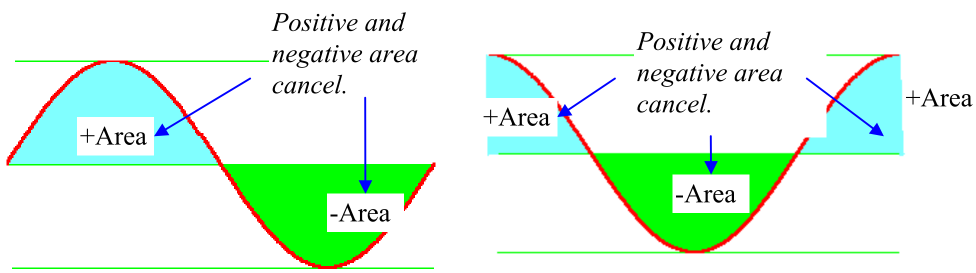


Figure 14 - The area under a sine or a cosine wave over one period is always zero.

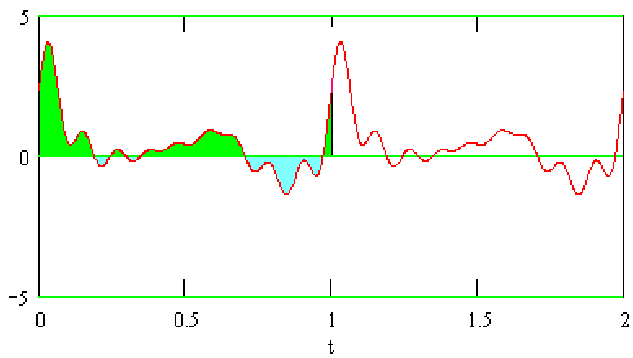
$$\int_o^T \left(\sum_{n=1}^{\infty} a_n \sin w_n t \right) dt = 0$$

$$\int_o^T \left(\sum_{n=1}^{\infty} a_n \sin w_n t + b_n \cos w_n t \right) dt = 0$$

The same is also true of the sum of sine and cosines. Any wave made up of sum of the sine and cosine waves also has zero area over one period. So we see that if we were to integrate our signal over one period the area obtained will have to come from coefficient a_0 only. The harmonics can make no contribution and they fall out.

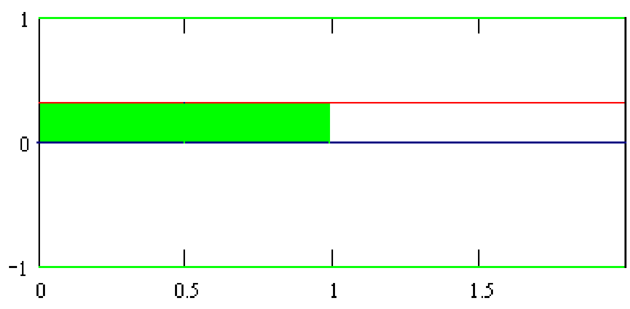
$$\int_0^T f(t)dt = \int_0^T a_o dt + \overbrace{\int_o^T \left(\sum_{n=1}^{\infty} a_n \sin w_n t + b_n \cos w_n t \right) dt}^{0} \qquad (8)$$

The second term is zero in (8), since it is just the integral of a wave made up of sine and cosines. Now we can compute a_0 by taking the integral of our complicated looking wave over one period.



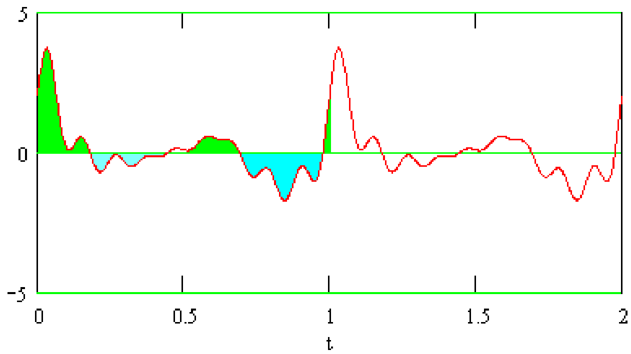
The wave has non-zero area in one period, which means it has a DC offset.

Figure 16 - Signal to be analyzed, looks like it has a dc offset since there is more area above the x-axis than below.



All area comes from the a_0 coefficient.

Figure 16a - The dc component



Area under the wave when shifted down is zero.

Figure 16b - Signal without the dc component

The area under one period of this wave is equal to

$$\int_0^T f(t)dt = \int_0^T a_o dt \qquad (9)$$

Integrating this very simple equation we get,

$$\int_0^T f(t)dt = a_0 T \qquad (10)$$

We can now write a very easy equation for computing a_0

$$a_0 = \frac{1}{T}\int_0^T f(t)dt \qquad (11)$$

Since no harmonics contribute to area, we see that a_0 is equal to simply the area under our complicated wave for one period divided by T, the integral period. We can compute this area in software and if it is zero, then there is no dc offset. This is also the mean value of the signal. A signal with zero mean value has no dc offset.

Computing a_n

Now we employ a slightly different trick from basic trigonometry to compute the coefficients of the sine waves. Here is a sine wave of an arbitrary frequency nω that has been multiplied by itself.

$$f(t) = \sin n\omega t * \sin n\, \omega t$$

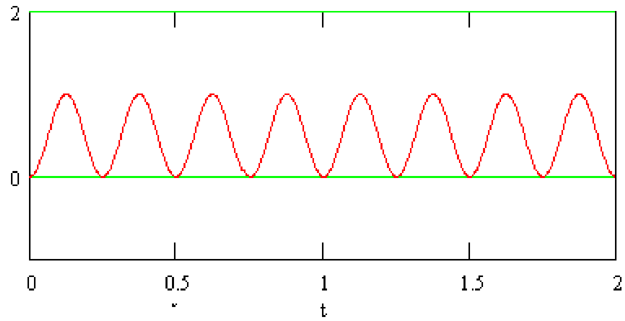


Figure 17 - The area under a sine wave multiplied by itself is always non-zero.

We notice that the resulting wave lies entirely above the x-axis and has a net positive area. From integral tables we can compute the area as equal to

$$\int_0^T a_n (\sin n\omega t)(\sin m\omega t)\, dt = a_n T/2 \quad for\ n = m \tag{12}$$

Where T is the period of the fundamental harmonic. But now let's multiply the sine wave by an arbitrary harmonic of itself to see what happens to the area.

$$f(t) = \sin n\,\omega t * \sin m\,\omega t$$

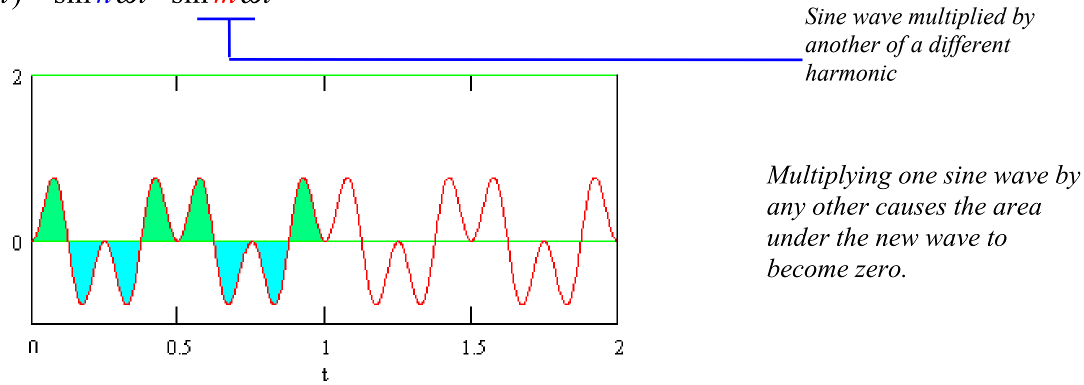


Figure 18 - The area under a sine wave multiplied by its own harmonic is always zero.

The area in one period of a sine wave multiplied by its own harmonic is zero. We conclude that when we multiply a signal by a particular harmonic, the only contribution comes from that particular harmonic. All others harmonics contribute nothing and fall out.

$$\int_0^T a_n (\sin n\omega t)\ (\sin m\omega t)\ dt = 0 \quad for\ n \neq m$$

(12)

$$\int_0^T a_n (\sin n\omega t)\ (\sin m\omega t)\ dt = a_n T/2 \quad for\ n = m$$

Now let's multiply a sine wave by a cosine wave to see what happens.

$f(t) = \sin n\omega t * \cos m\omega t$

Sine wave multiplied by a cosine wave for any n and m

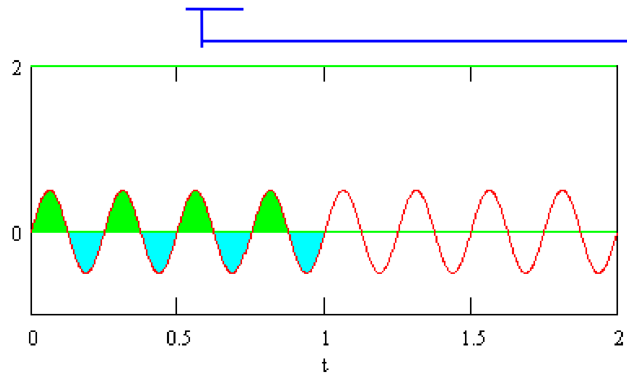


Figure 19 - The area under a cosine wave multiplied by a sine wave is always zero.

It seems that the area under the wave which is multiplication of a sine and cosine wave is always zero whether the harmonics are the same or not. Summarizing, by setting $\omega_n = n\omega$

$$\int_0^T a_n (\sin \omega_n t)\ (\sin \omega_m t)\ dt = 0 \quad for\ n \neq m$$

$$\int_0^T a_n (\sin \omega_n t)\ (\sin \omega_m t)\ dt = a_n T/2 \quad for\ n = m \qquad (13)$$

$$\int_0^T a_n (\cos \omega_n t)\ (\sin \omega_m t)\ dt = 0 \quad for\ all\ n\ and\ m$$

Rules:

1. The area under one period of a sine or a cosine is zero.

2. The area under one period of a wave that is a product of two sine or cosine waves of non-harmonic frequencies is zero.

3. The area under one period of a wave that is a product of two sine or cosine waves of same harmonic frequency is non-zero and not equal to $a_n T/2$, where T is the period of the resolution frequency we have chosen.

4. The area under one period of a wave that is a product of a sine wave and a cosine wave of any frequencies (different or equal) is equal to zero.

Recall that in vector representation, sine and cosines are orthogonal to each other. So all harmonics are by definition orthogonal to each other.

A very satisfying interpretation of the above rules is that sine and cosine waves can act as filtering signals. In essence they act as narrow-band filters and take out all frequencies except the one of interest. This forms the basic concept of a filter.

Now let's use this information. Successively multiply the Fourier equation by a sine wave of a particular harmonic and integrate over one period as in equation below.

$$\int_o^T f(t)\sin(nwt)\,dt = \overbrace{\int_0^T a_0 \sin(wt)\,dt}^{0} + \int_0^T a_n \sin(n\omega t)\sin(n\omega t)\,dt + \overbrace{\int_0^T b_n \cos(n\omega t)\sin(n\omega t)\,dt}^{0}$$

We know that the integral of the first and the third term is zero since the first term is the integral of a sine wave multiplied by a constant (Rule 1) and the third is a sine wave multiplied by a cosine wave (Rule 3). This simplifies our equation considerably. The integral of the second term is

$$\int_0^T a_n \sin(n\omega t)\sin(n\omega t)\,dt = \frac{a_n T}{2} \qquad (13)$$

From this we write the equation to obtain a_n, which are the coefficients of each of the sine waves as follows

$$a_n = \frac{2}{T}\int_0^T f(t)\sin(n\,\omega t)\,dt \qquad (14)$$

The a_n is then computed by taking the signal over one period, successively multiplying it with a sine wave of n times the starting fundamental frequency and then integrating. This gives the coefficient for that particular harmonic.

Imagine we have a signal that consists of just one frequency, we think it is around 5 Hz (and is a sine wave from). We begin by multiplying this signal by a sine wave of frequency .2 and each of its harmonics which are .4, .6, .8 ,…..10 and so on. Actually since we know it is in the range of 5 Hz, we can dispense with the lower harmonics say up to 4 and start with 4.2 and go to 5.8 Hz.

Here is all the math we do.

1. Multiple the wave with a sine wave of frequencies 4.2 and integrate the result. Most likely the result will be zero.
2. Go to next harmonic, which 4.4. This is 22nd harmonic of the resolution frequency .2 Hz.
3. Repeat step 1 and 2 and continue until harmonic frequency is equal to 5.8 Hz.

The results will show that the integrals of all harmonics frequencies are zero, except for the 25th harmonic, the integral of which will be equal to

$$= \frac{a_{25}T}{2} = 2.5a_{25}$$

$$a_{25} = \frac{\text{One period integral}}{2.5}$$

Where T = 1/f = 1/.2 = 5 sec. The coefficient can now be calculated which gives the amplitude of the wave. (We already know its frequency, which is 5 Hz, since the integral is non-zero for that component.).

Computing coefficient of cosines, b_n

Now instead of multiplying by a sine wave we multiply by a cosine wave. The process is exactly the same as above.

$$\int_o^T f(t)\cos(n\omega t)\,dt = \overbrace{\int_0^T a_0\cos(n\omega t)\,dt}^{0} + \overbrace{\int_0^T a_n\sin(n\omega t)\cos(n\omega t)\,dt}^{0} + \int_0^T b_n\cos(n\omega t)\cos(n\omega t)\,dt$$

Now terms 1 and 2 become zero. (First term is zero from rule 1, the second term due to rule 3.) The third terms is equal to

$$\int_0^T b_n\cos(n\omega t)\cos(n\omega t)\,dt = \frac{b_nT}{2} \qquad (14)$$

and the equation can be written as

$$b_n = \frac{2}{T}\int_0^T f(t)\cos\left(n\omega t\right) dt \qquad (15)$$

So the process of finding the coefficients is multiplying our signal with successively larger frequencies of a fundamental wave and integrating the results. This is easy to do in software. The results obtained successively are the coefficient for each frequency of the harmonic wave. We do the same thing for sines and cosine coefficients.

Following this process, we compute the coefficients of the following wave

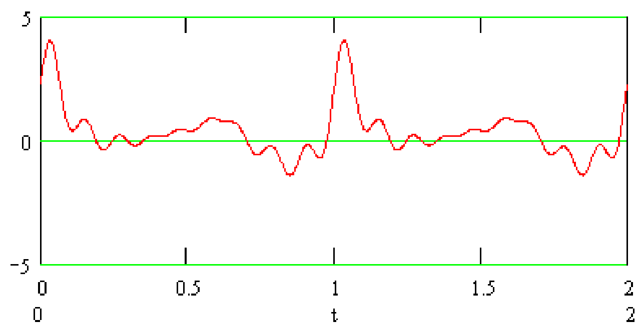


Figure 20 - The signal to be analyzed

Without going through the math, we will give the answers in two vectors, first is the coefficients of the sine and second the cosine waves and the dcoffset.

a_n = [.4 .3 .7 .3 .3 .3 .2 .3 .4]

b_n = [.05 .2 .7 .5 .2 .2 .1 .05 .02]

a_0 = .32

From this we can write the equation of the above wave as

$$\begin{aligned} f(t) = {} & .32 + .4\sin(2\pi 1)t + .3\sin(4\pi)t + .7\sin(6\pi)t \\ & + .3\sin(8\pi)t + .3\sin(10\pi)t + .3\sin(12\pi)t + .2\sin(14\pi)t + ... \\ & + .05\cos(2\pi 1)t + .2\cos(4\pi)t + .7\cos(6\pi)t + .5\cos(8\pi)t + \\ & + .2\cos(10\pi)t + .2\cos(12\pi)t + .1\cos(14\pi)t + ... \end{aligned}$$

The coefficients are the amplitudes of each of the harmonics. The resolution frequency is 1 Hz and the harmonics are integer multiples of this frequency. Now we know exactly what the components of the received wave are. If the transmitted wave consisted only of one of these frequencies, then, we can filter this wave and get back the transmitted signal.

Summary

The Fourier series is given by

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \sin(\omega_n t) + \sum_{n=1}^{\infty} b_n \cos(\omega_n t)$$

where

$$\omega_n = 2\pi n f$$

The coefficients of the Fourier series are given by

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\sin\left(n\omega t\right)dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\cos\left(n\omega t\right)dt$$

where ω is the fundamental frequency and is related to T by

$$\omega_n = 2\pi f_n = 2\pi \frac{n}{T}$$

Coefficients become the spectrum

Now that we have the coefficients, we can plot the magnitude spectrum of the signal.

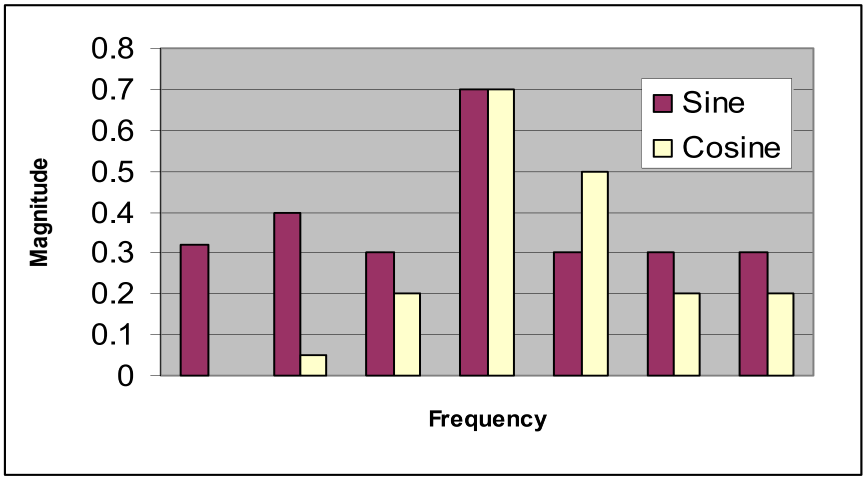


Figure 21 - The Fourier series coefficients for each harmonic

You may now say that this spectrum is in terms of sines and cosines, and this is not the way we see it in books. The spectrum ought to give just one number for each frequency.

We can compute that one number by knowing that most signal are represented in complex notation where sine and cosine waves are related in quadrature. The total power shown on the y axis of the spectrum is the power in both the sine and cosine waves in the real and imaginary components of the same frequency. We can compute the magnitude by from the root sum square of the sine and cosine coefficients for each harmonic including the dc offset of the zero frequency value.

$$Magnitude = \sqrt{a_n^2 + b_n^2}$$

Plot the modified spectrum

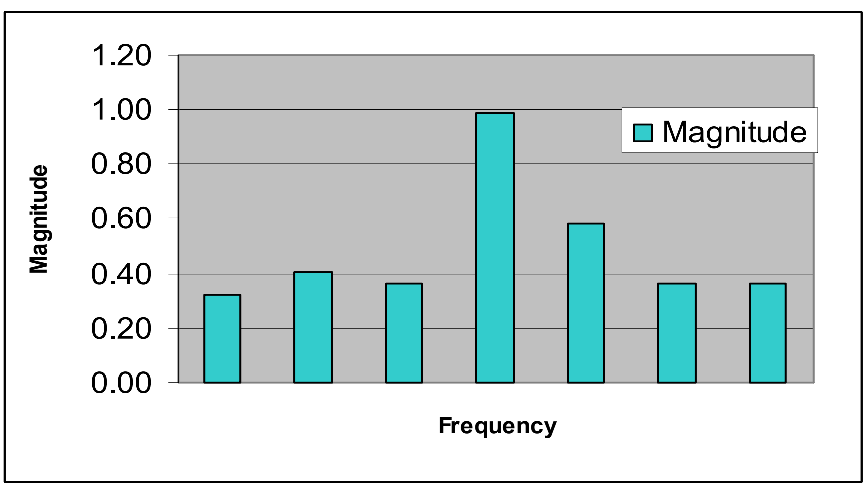


Figure 22 - A traditional looking spectrum created from the Fourier coefficients

Voila! Although this is not a real signal, we see that it now looks like a traditional spectrum. The largest component is at frequency = 3. The y-axis can easily be converted to dB. In complex representation, the phase of the signal is defined by

$$\phi_n = \tan^{-1}(b_n / a_n)$$

For every frequency, we can also compute and plot the phase. Phase plays a very important role in signal processing and particularly in complex representation and shows useful information about the signal.

One thing you may not have noticed during this computation of the coefficients is that they will be different depending on what you pick as the resolution frequency. We will get different answers depending on the choice we make for this number. In essence depending on the resolution, the signal energy leaks from one frequency to the next so we get different answers, but the overall picture remains the same. The issues of leakage will discussed later.

We also stated that the wave has to be periodic. But for real signals we can never tell where the period is. Random signals do not have discernible periods. In fact, a real signal may not be periodic at all. In this case, the theory allows us to extend the "period" to infinity so we just pick any representative section of our signal or even the whole signal and call it "The Period". Mathematically this assumption works out just fine for real signals.

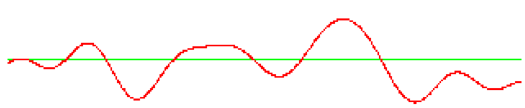


Figure 23 - We call the signal periodic, even though we don't know what lies at each end.

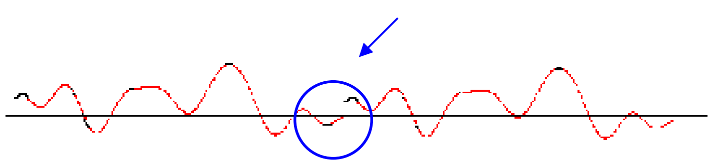


Figure 24 - Our signal repeated to make it mathematically periodic, but ends do not connect and have discontinuity

The part of the signal that we pick as representing the real period is only a sample of the whole and not really the actual period. The end section of the chosen section will most likely not match as they would for a real periodic signal. The error introduced into our analysis due to this end mismatch is called aliasing. Windowing functions are used to artificially shape the ends so that they are zero at the ends and so the chosen signal portion is made artificially periodic. This introduces errors in the analysis which have to be dealt with by other techniques.

Next the complex representation.

I can be reached at
mntcastle@earthlink.net

Other tutorials at
www.complextoreal.com