Lecture 4 – Analogue Realisation of Filter Transfer Functions

4.1 LCR filters

The use of all 3 types of linear electric circuit elements – R's, L's and C's – enables poles or zeroes to be placed anywhere in the s-plane, and in particular anywhere to the left of the imaginary axis; hence any Butterworth, Chebyshev, elliptic or Bessel-Thomson filter can in theory be realised with a passive LCR filter. The synthesis of such filters is a subject on its own and we will only cover the basics in this lecture.

Although good band-pass characteristics can be achieved with passive LCR filters, the use of inductors is best avoided, especially at low frequencies. Inductors tend to be lossy (i.e. they have significant series resistance), expensive and bulky over the low-frequency range. The use of active devices allows inductors to be eliminated from the network, as a circuit with R's and C's only but *with gain* can have poles or zeroes anywhere in the s-domain.

4.2 Active filters

These are usually built around operational amplifiers which have a high input impedance (i.e. a JFET input stage). Component accuracy better than 10% is required, and 5% or even 2% may be needed for Chebyshev and higher-order filters. There are a number of circuit designs, and you will already have met some of these in the Electronic Instrumentation course in your second year. Some of what follows should therefore be revision.

4.2.1 Low-pass filters

The general transfer function for a second-order low-pass filter is :

$$G(s) = \frac{K}{1 \;+\; a_1(s/\omega_c) \;+\; a_2(s/\omega_c)^2}$$

where K = d.c. gain of filter and ω_c = cut-off frequency The circuit in Fig. 4.1 will implement the above transfer function. To analyse the circuit, start from the

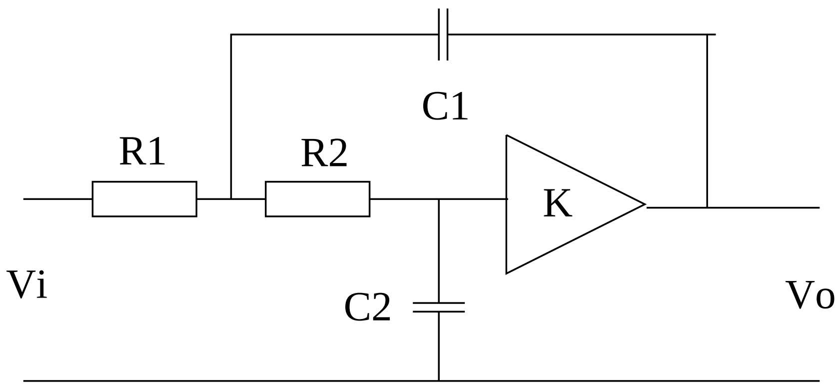

Figure 4.1: *Second-order LPF circuit.*

right and work towards the left; the final result (see Second Year lecture notes) is:

$$\frac{v_o}{v_i}(s) = \frac{K}{1 + s[C_2(R_1 + R_2) \;+\; (1-K)C_1R_1] + \; s^2C_1C_2R_1R_2}$$

For a Butterworth filter, a_2 = 1 and a_1 = 2ζ

$$\text{i.e.} \quad \omega_c = \frac{1}{\sqrt{C_1C_2R_1R_2}} \qquad\qquad \zeta = \sqrt{\frac{C_2}{C_1}} \cdot \frac{R_1+R_2}{2\sqrt{R_1R_2}} - \frac{(K-1)}{2}\sqrt{\frac{C_1R_1}{C_2R_2}}$$

To implement the Butterworth second-order section, there are two possibilities:

Sallen-Key filter ($K = 1$)

If $K = 1$, then $\zeta = \frac{1}{2}\sqrt{\frac{C_2}{C_1}} \cdot \frac{R_1+R_2}{\sqrt{R_1R_2}}$ which is greater than $\sqrt{\frac{C_2}{C_1}}$ since $\frac{R_1+R_2}{2} > \sqrt{R_1R_2}$.

So for $\zeta < 1$, we need $C_1 > C_2$. Usually $R_1 = R_2 = R$ and the resistor value is chosen to be as large as possible so that the capacitors will be small (in general, the capacitors will have to be made from a combination of individual components).

VCVS filter ($K > 1$)

An ideal voltage–controlled voltage source (VCVS) is a voltage amplifier with $Z_{in} = \infty$, $Z_{out} = 0$ and a constant voltage gain. The high input impedance, low output impedance and stable gain of an op-amp configured as a non-inverting amplifier make it a good approximation to an ideal VCVS. With the VCVS circuit,

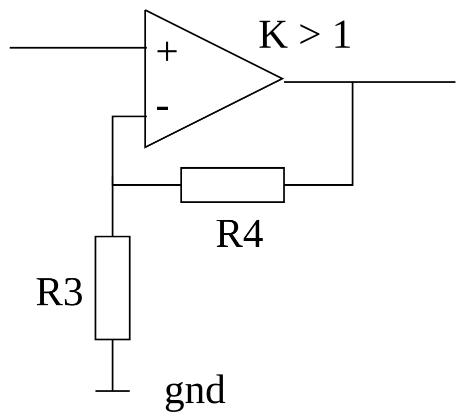

Figure 4.2: $K > 1$ *gain stage for VCVS filter.*

we can now set $C_1 = C_2$ and $R_1 = R_2$. The disadvantage is that the dc gain is constrained to be some odd value, difficult to obtain precisely unless close–tolerance resistors are used for R_3 and R_4.

4.3 High–pass filters

$$G(s) = \frac{K(s/\omega_c)^2}{1 + 2\zeta(s/\omega_c) + (s/\omega_c)^2}$$

where K = high–frequency gain[4] and ω_c = cut–off frequency

[4] in practice, the response of the high–pass filter must fall off at high frequencies because of the open–loop bandwidth limitations of the op–amp.

Such a filter can be realised simply by interchanging the resistors and capacitors in the circuit of Fig. 4.1.

4.4 Band-pass filters

$$G(s) = \frac{K \, . \, 2\zeta(s/\omega_n)}{1 + 2\zeta(s/\omega_n) + (s/\omega_n)^2}$$

where K = mid-band gain and ω_n = centre frequency.

The above is normally re–written as :

$$G(s) = \frac{(K/Q) \, . \, (s/\omega_n)}{1 + \frac{1}{Q}(\frac{s}{\omega_n}) + (\frac{s}{\omega_n})^2}$$

since Q = quality factor = $1/2\zeta$ = $\omega_n/\Delta\omega$

$\Delta\omega$ is the 3 dB bandwidth of the bandpass filter, ie. $\omega_u - \omega_l$ where ω_u is the upper limit of the passband and ω_l the lower limit.

The band-pass version of the VCVS filter circuit (see Fig. 4.3) can be analysed in the same way as the low-pass filter circuit, although one ends up with very cumbersome expressions for K, ω_n and Q (see G.B. Clayton, *Linear Integrated Circuit Applications*, page 63). The multiple-feedback design that follows is a much more popular alternative for band-pass filters.

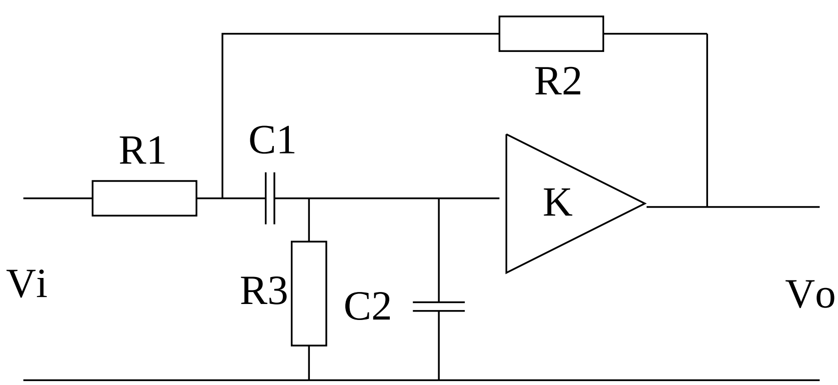

Figure 4.3: *VCVS band-pass filter circuit*

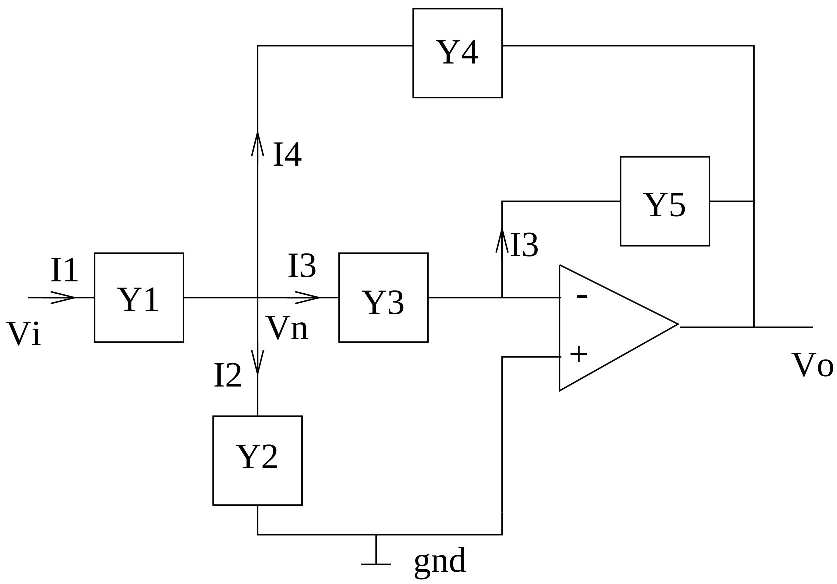

Figure 4.4: *Generalised multiple feedback circuit.*

4.5 Multiple feedback circuits

Summing the currents at node N gives:

$$I_1 = Y_1(v_i - v_N) = I_2 + I_3 + I_4 = Y_2 v_N \ + \ Y_3 v_N \ + \ Y_4(v_N - v_o)$$

Re-arranging the above leads to:

$$Y_1 v_i = v_N(Y_1 + Y_2 + Y_3 + Y_4) \ - \ Y_4 v_o$$

We also have:

$$Y_3 v_N \ = -Y_5 v_o \quad \rightarrow \quad v_N = -\frac{Y_5}{Y_3} v_o$$

$$Y_1 Y_3 v_i = v_o[-Y_5(Y_1 + Y_2 + Y_3 + Y_4) \ - \ Y_3 Y_4]$$

and so we end up with:

$$\frac{v_o}{v_i} = \frac{-Y_1 Y_3}{Y_5(Y_1 + Y_2 + Y_3 + Y_4)\ +\ Y_3 Y_4}$$

For a low–pass filter, $Y_1 = 1/R_1; Y_3 = 1/R_2; Y_4 = 1/R_3$ and $Y_2 = sC_1; Y_5 = sC_2$.

For a high–pass filter, swap the R's and C's, as before.

For a band–pass filter, $Y_1 = 1/R_1; Y_2 = 1/R_2; Y_5 = 1/R_3$ and $Y_3 = sC_1; Y_4 = sC_2$.

Example of a band-pass filter using multiple feedback design

For the case when $C_1 = C_2$, we have the following circuit,

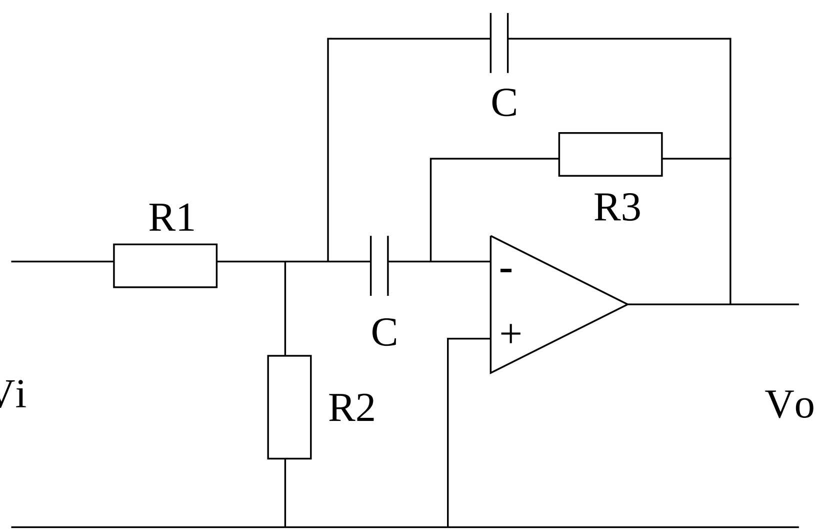

Figure 4.5: *Multiple feedback 2nd-order band-pass filter circuit.*

$K = -\frac{1}{2}\frac{R_3}{R_1}$ (gain at resonance)

$\omega_n = \frac{1}{C\sqrt{R_{12}.R_3}}$

$Q = 0.5\sqrt{R_3/R_{12}}$

where $R_{12} = \frac{R_1 R_2}{R_1 + R_2}$.

Since $Q^2 = \frac{1}{4}\frac{R_3}{R_{12}}$ and $\mid \frac{K}{2} \mid = \frac{1}{4}\frac{R_3}{R_1}$, we must have $Q^2 > \mid \frac{K}{2} \mid$.

Note that, since Q depends on the square root of component ratios, a high-Q filter (≈ 100) requires very high component ratios. In practice, R_2 needs to be adjustable in order to get the correct resonant frequency, as the latter is affected by the non-ideal behaviour of the op-amp.

4.6 State-variable filters

The performance of second-order sections can be improved by the introduction of additional op-amps. The disadvantage of increased power consumption is more than offset by reduced sensitivity to component variations and the use of a standard topology to realize the basic frequency responses.

One such circuit is the state-variable filter which can provide second-order low-pass, band-pass and high-pass outputs simultaneously. The circuit consists of two integrators and an inverting gain stage. In the version shown below, negative feedback around all 3 stages is provided by R_5, whilst R_6 and R_7 form a positive feedback loop around the first two stages. It is a simple matter to show that v_A, v_B and v_C are the high-pass, band-pass and low-pass outputs, respectively.

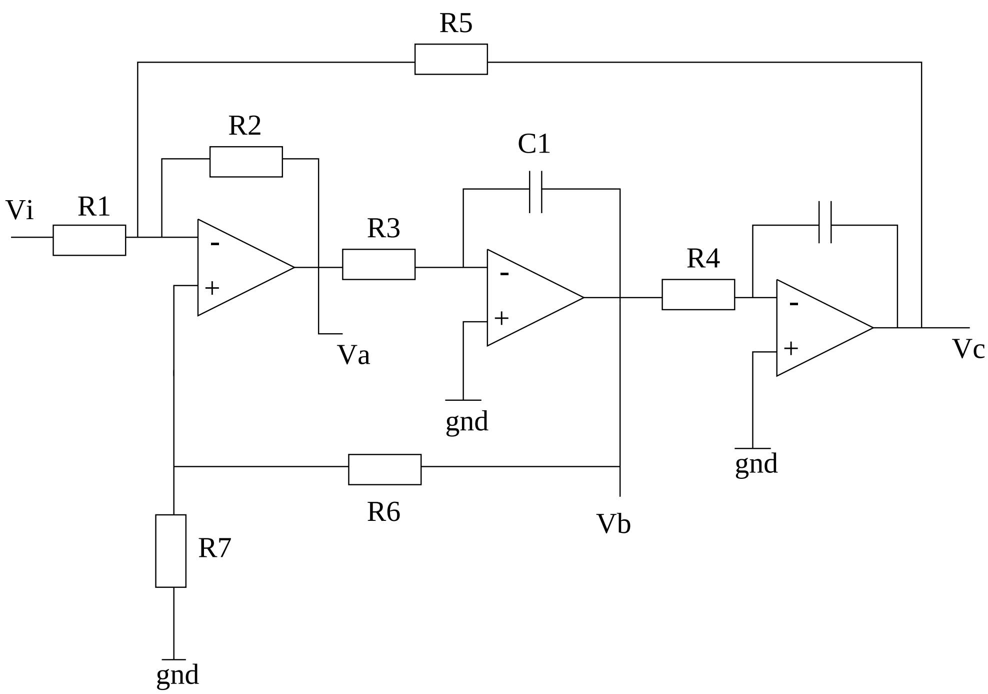

Figure 4.6: *Universal or state-variable filter.*

4.7 Switched-capacitor filters

Switched-capacitor technology first arose from the need to implement analogue filters as integrated circuits. It is relatively easy to fabricate switches, capacitors and op-amps in VLSI, but not resistors. The problem was solved by approximating resistors using two MOS switches and a capacitor, as will now be explained.

4.7.1 The switched capacitor

The key to the circuit in Fig. 4.7 operating as a resistor is the use of two *anti-phase* clocks ϕ_1 and ϕ_2 so that, when MOSFET A is on, then B is off and vice-versa (i.e. a single-pole double-throw switch). Assume intitially that A is on and B is off.

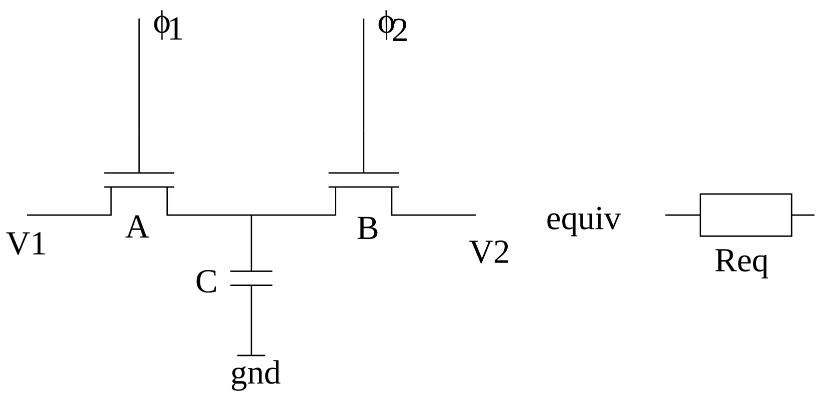

Figure 4.7: *Equivalent resistor.*

Then capacitor C will charge up to voltage v_1 with a time constant equal to CR_{DS}, where R_{DS} is the on-resistance of the FET switch (this time constant needs to be small compared with variations in $v_1(t)$). On the next clock edge, the switch is changed to the other position (A off and B on) and the capacitor will discharge to v_2, provided that $v_2 < v_1$. Thus charge $C(V_1 - v_2)$ is transferred from left to right at every cycle. If the clock frequency is f_c, then the average current is:

$$i(t) = \frac{\Delta q}{\Delta t} = \frac{C(v_1 - v_2)}{1/f_c}$$

The size of an equivalent resistor to give the same value of current would be:

$$R_{eq} = \frac{v_1 - v_2}{i} = \frac{1}{f_c C}$$

Thus the switched capacitor is approximately equivalent to a resistor R_{eq} of value $1/(f_c C)$, provided that the switching frequency f_c is much larger than the highest frequency in $v_1(t)$ and $v_2(t)$. A typical value for f_c is 100 kHz, which means that switched-capacitor filters can be used for audio filters and in speech coders.

4.7.2 Switched-capacitor integrator

Consider the integrator (or first-order low-pass filter) circuit shown in Fig. 4.8, for which the transfer function is:

$$\frac{v_o}{v_i} = -\frac{1}{sC_2R}$$

The switched capacitor implementation has a transfer function given by substitut-

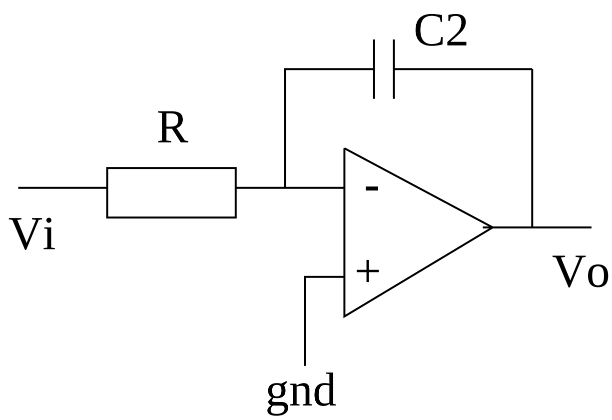

Figure 4.8: *Switched-capacitor integrator circuit.*

ing R_{eq} for R, such that:

$$\frac{v_o}{v_i} = -f_c \frac{C_1}{C_2} \frac{1}{s}$$

which involves the ratio $\frac{C_1}{C_2}$. Integrated MOS capacitors have a value which is determined by the dielectric constant, the thickness of the dielectric and the area of the capacitor. Assuming that the dielectric constant and thickness do not vary, the ratio of two capacitors made within the same integrated circuit will depend *only on their area ratio*. This is primarily determined by the geometrical shape of the capacitors which can be controlled accurately when using photolithographic techniques. Hence the *ratio* of capacitances can be realised with accuracy even though the *value* of capacitance cannot be.

Provided that $C_1 \ll C_2$ and, as before, that $f_c \gg$ pass-band frequencies, then the switched-capacitor implementation is an adequate approximation to a continuous time analogue integrator[5]. The time constant of the integrator ($\tau = \frac{C_2}{f_c C_1}$) – or 3dB bandwidth of the low-pass filter – can be accurately defined with a high degree of stability since it depends on a ratio of capacitor values; furthermore, it can be varied simply by adjusting the clock frequency.

4.7.3 Gain stages

The inverting op-amp amplifier can be converted to its switched-capacitor equivalent by replacing each resistor with a switched capacitor (see Fig 4.9). Both switch pairs are operated at the same clock frequency and the transfer function of the circuit is therefore given by:

$$\frac{v_o}{v_i} = -\frac{C_1}{C_2}$$

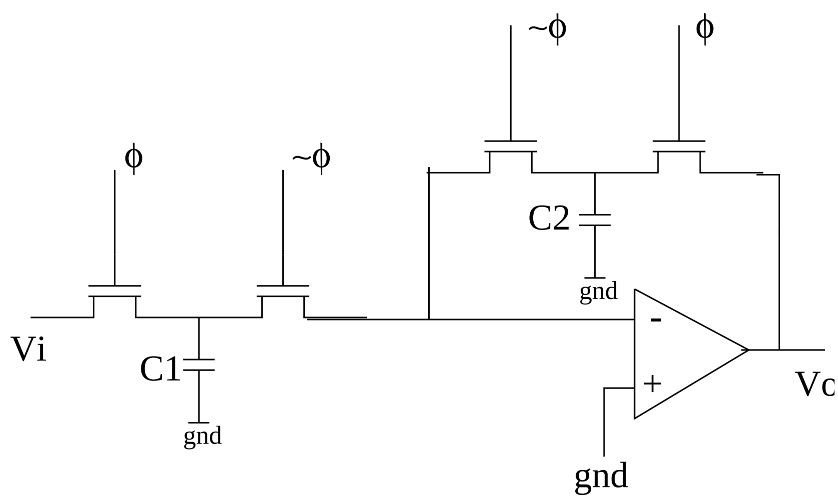

Figure 4.9: *First attempt at switched-capacitor inverting op-amp amplifier.*

[5]When the switching and signal frequencies are of the same order of magnitude, we cannot ignore the time sampling of the signal and sampled data techniques are required for analysis; we have to treat switched-capacitor filters as *digital* filters and use the z-transform – see next lecture.

As with the integrator, the gain depends on the *ratio* of two capacitors. The above circuit, however, is impratical. Since the FET switches used to implement the feedback resistor are never closed simultaneously, no feedback around the op-amp is provided. For this reason, the circuit below is used instead. When $\phi = 1$, C_1 is charged to the input signal v_i and C_2 is discharged (so that the charge previously stored on the previous cycle is not retained). When $\phi = 0$, the voltage on C_1 is applied to the op-amp and C_2 forms the feedback path.

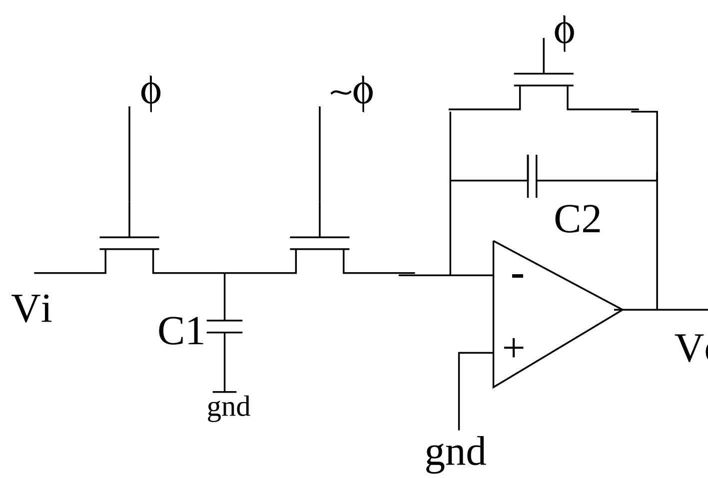

Figure 4.10: *Practical switched-capacitor inverting op-amp amplifier.*

4.7.4 Switched-capacitor second-order sections

There are two main constraints which conversion to an equivalent switched-capacitor implementation imposes on these circuits:

- *switched-capacitor resistors cannot close an op-amp feedback path*. The solution of the previous page is not applicable to all circuits.
- *no floating nodes*. All capacitive plates are subject to charge accumulation from a variety of parasitic sources such as leakage currents. In order to insure stability of the circuit, there must be a path either directly or through switched-capacitor resistors from every node in the circuit to a voltage source. Circuits with floating nodes, such as those with two passive

elements in series, have to be avoided and this rules out Sallen-Key circuits, for example.

Of all the second-order sections considered in the last few pages, only the state-variable filter is ideally suited to switched-capacitor implementation (and indeed is available commercially as a programmable analogue filter chip).

4.8 Higher-order filters

The most widely used synthesis procedure for realising high-order filters is to cascade $n/2$ second-order sections, if n is even, and $(n-1)/2$ second-order sections together with one first-order section if n is odd. In general, the individual second-order sections in a cascade filter are not identical; **each section represents a quadratic polynomial factor in the n-th order polynomial describing the overall filter**. Fortunately, this problem has been solved many times over and there are design equations and tables in most filter handbooks for all the standard filter responses. There are tables for both the unity-gain Sallen-Key and VCVS filters, although the latter is more popular for cascaded stages: each stage has gain so that, as the bandwidth is reduced through the filter, the full bandwidth rms signal is kept nearly constant. This helps to prevent the introduction of significant amplifier noise and allows large signals to be used.