Solving Linear Equations

Vectors and Linear Equations

Let's start with a simple linear system with two equations and 2 unknowns:

$$\begin{array}{rcrcr} x & - & 2y & = & 1 \\ 3x & + & 2y & = & 11 \end{array}$$

The system can be interpreted in 3 ways:

Row picture: We interpret one row at a time, each representing a straight-line equation in xy plane.

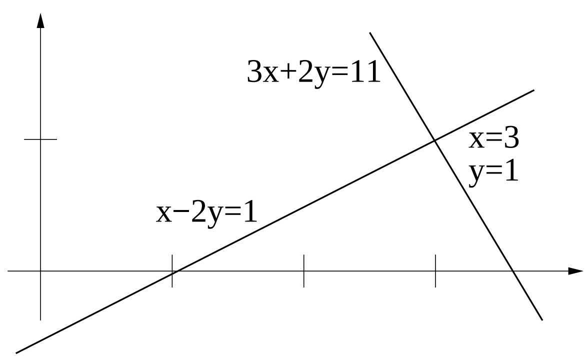

Figure 1: Row Picture: The intersection point is the solution.

- The intersection point, (3,1), lies on both lines.
- In other words, point (3,1) satisfies both equations and is the solution

Column picture: We treat each column as a *vector*, and view the system as a "vector equation".

$$x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} = b$$

- What we want is to *find the combination of the LHS vectors equals to the RHS vector*.
- The vector equation is a *linear combination* of the column vectors.

$$3 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

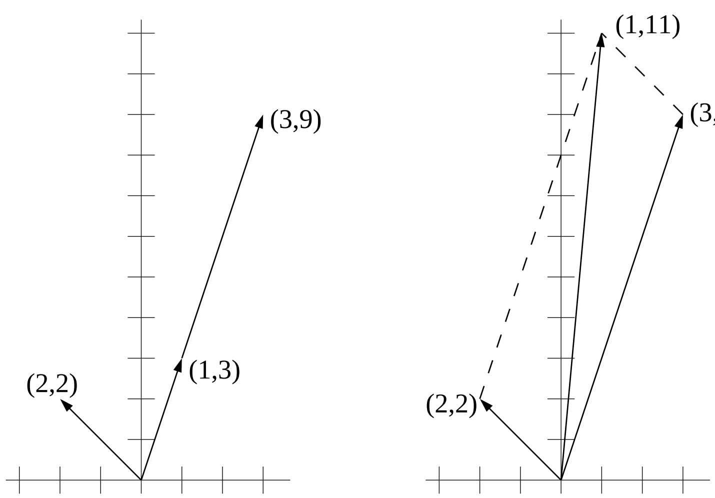

Figure 2: Column Picture: A combination of columns produces the RHS.

Exercise1: Let's consider the row and column pictures of the linear system with 3 unknowns, x, y, and z.

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 6 \\ 2x & + & 5y & + & 2z & = & 4 \\ 6x & - & 3y & + & z & = & 2 \end{array}$$

Can you visualize:

- Row picture shows 3 planes meeting at a single point.
- Column picture combines 3 columns to produce the vector.

Matrix picture: We separate the coefficients from the unknowns, and form a "coefficient matrix" $\mathbf{A}$. The system becomes $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

To visualize $\mathbf{A}\boldsymbol{x}$:

By rows: dot-products of rows with $\boldsymbol{x}$

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}.\boldsymbol{x} \\ \\ \begin{bmatrix} 2 & 5 & 2 \end{bmatrix}.\boldsymbol{x} \\ \\ \begin{bmatrix} 6 & -3 & 1 \end{bmatrix}.\boldsymbol{x} \end{bmatrix}$$

By Columns: $\mathbf{A}\boldsymbol{x}$ is a combination of the column vectors.

$$\mathbf{A}\boldsymbol{x} = x \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + y \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + z \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Question1: What is the solution $\boldsymbol{x}$ if we use a different $\boldsymbol{b} = (3, 7, 3)$?
Question2: Can we solve $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ for every $\boldsymbol{b}$?

Gaussian Elimination

- Elimination was introduced before (in junior high-school, maybe?)

$$\begin{array}{rcrcr} x & - & 2y & = & 1 \\ 3x & + & 2y & = & 11 \end{array}$$

  What is the system after elimination?

- Some terminology:

  Forward elimination: The process that generates an upper triangular system.

  Back substitution: Upper triangular systems are solved in reverse order.

  Pivot: First **nonzero** in the row that does the elimination.

  Multiplier: (*entry to eliminate*) divided by (*pivot*)

- Let's do the Gaussian elimination on a 3 by 3 system:

$$\begin{array}{rcrcrcr} 2x & + & 4y & - & 2z & = & 2 \\ 4x & + & 9y & - & 3z & = & 3 \\ -2x & - & 3y & + & 7z & = & 10 \end{array}$$

- What can go wrong?

  No solution:

$$\begin{array}{rcrcr} x & - & 2y & = & 1 \\ 3x & - & 6y & = & 11 \end{array}$$

  Infinitely many solutions:

$$\begin{array}{rcrcr} x & - & 2y & = & 1 \\ 3x & - & 6y & = & 3 \end{array}$$

  Zero pivot:

$$\begin{array}{rcrcr} 0x & + & 2y & = & 4 \\ 3x & - & 2y & = & 5 \end{array}$$

- The first and second scenarios are **singular**.

- We want to combine two ideas — elimination and matrices, i.e., we create zeros below the pivot using an elimination matrix.

- Back to the previous example, we want to find the two elimination matrices $\mathbf{E_1}$, and $\mathbf{E_2}$, such that $\mathbf{E_2 E_1 A}$ is the upper triangular matrix of forward elimination.

- The elimination process generates $\mathbf{E_2 E_1}(\mathbf{A}\boldsymbol{x}) = \mathbf{E_2 E_1}\boldsymbol{b}$. This is also $(\mathbf{E_2 E_1 A})\boldsymbol{x} = \mathbf{E_2 E_1}\boldsymbol{b}$.

In-Class Exercise

Consider the system of equations:

$$\begin{array}{rcrcrcl} x & + & 3y & + & 5z & = & 4 \\ x & + & 2y & - & 3z & = & 5 \\ 2x & + & 5y & + & 2z & = & 8 \end{array}$$

- Looking at the row picture, are any two of the planes parallel? What are the equations of planes parallel to $x + 3y + 5z = 4$?

- Does the system have a solution?

- Can you modify one equation such that the system gives infinitely many solutions?

- Find 3 RHS vectors (i.e. 3 different values for b) that allow solutions.

Exercises

- Selected problem from Problem Set 2.1: 1, 2, 3, 6, 10, 11, 21, 23, 27, and 28.

- Selected problem from Problem Set 2.2: 1, 2, 5, 11, 13, 18, 20, 25.