Harvey Mudd College Math Tutorial:

Limit Definition of the Derivative

Once we know the most basic differentiation formulas and rules, we compute new derivatives using what we already know. We rarely think back to where the basic formulas and rules originated.

The geometric meaning of the derivative

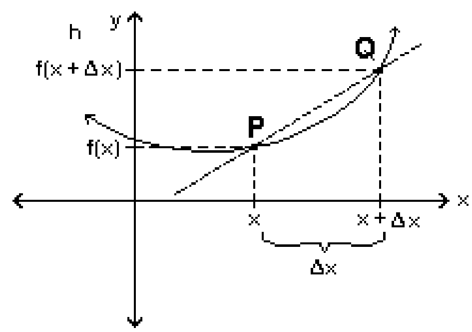
$$f'(x) = \frac{df(x)}{dx}$$

is the slope of the line tangent to $y = f(x)$ at x.

Let's look for this slope at P:

The **secant** line through P and Q has slope

$$\frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$



We can approximate the **tangent** line through P by moving Q towards P, decreasing Δx. In the limit as $\Delta x \to 0$, we get the tangent line through P with slope

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

We define

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}^{*}.$$

* If the limit as $\Delta x \to 0$ at a particular point does not exist, $f'(x)$ is undefined at that point.

We derive all the basic differentiation formulas using this definition.

Example

For $f(x) = x^2$,

$$f'(x) \quad = \quad \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$
\begin{aligned}
&= \lim_{\Delta x \to 0} \frac{(x^2 + 2(\Delta x)x + \Delta x^2) - x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2(\Delta x)x + \Delta x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} (2x + \Delta x) \\
&= 2x
\end{aligned}
$$

as expected.

Example

For $f(x) = \dfrac{1}{x}$

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{x-(x+\Delta x)}{(x+\Delta x)(x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{-\Delta x}{(x+\Delta x)(x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{-1}{(x+\Delta x)(x)} \\
&= -\frac{1}{x^2}
\end{aligned}
$$

again as expected.

Notes

The limit definition of the derivative is used to prove many well-known results, including the following:

- If f is differentiable at x_0, then f is continuous at x_0.
- Differentiation of polynomials: $\dfrac{d}{dx}\left[x^n\right] = nx^{n-1}$.
- Product and Quotient Rules for differentiation.

Key Concepts

We define $f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$.

[I'm ready to take the quiz.] [I need to review more.]
[Take me back to the Tutorial Page]