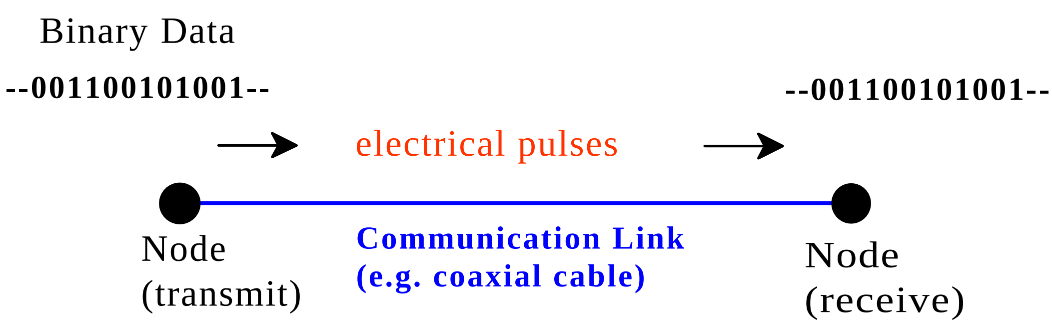
TCOM 370 NOTES 99-2

FOURIER SERIES, BANDWIDTH, AND SIGNALING RATES ON DATA TRANSMISSION LINKS

1. PULSE TRANSMISSION

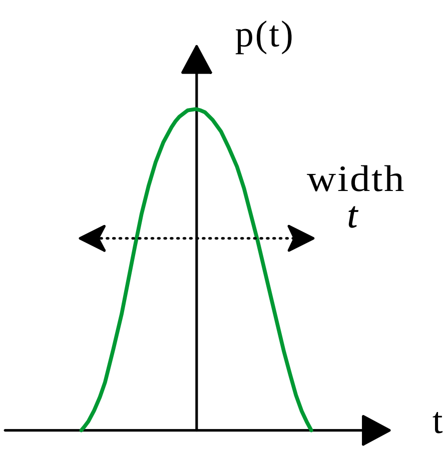
We now consider some fundamental aspects of binary data transmission over an individual communication link between two nodes.



In a simple and practical scheme, each binary digit (**0** or **1**) is transmitted as one of two possible electrical pulse signaling waveforms:

0 → sent as $p_0(t)$

1 → sent as $p_1(t)$



In particular, we may use a ***single*** pulse waveform p(t) to transmit the binary digits, using two ***different amplitudes*** for the different digits.

A simple choice is to use a positive amplitude and a negative amplitude, so that $p_1(t) = p(t)$ and $p_0(t) = -p(t)$ ("bipolar pulses")

Key Points:

1. If the width of each signaling pulse is t (seconds), then we may transmit data at a rate of $\frac{1}{t}$ bits per second (bps).

2. The narrower the pulse, the higher the bit rate.

There is a limit as to how narrow we can make the pulses for any given physical link; this limit is determined by a characteristic called the "bandwidth" of the communication link.

Definition:

The **bandwidth** of a link is the **width of the band of sinewave frequencies** (in cycles per second, or Hz) that can be carried by the link with reasonable fidelity and not be grossly attenuated during propagation.

3. The larger the bandwidth, the narrower the pulses that may be propagated on the link.

4. Available link bandwidth depends on the physical/electrical characteristics of the link, as well as on the limitations imposed by any electrical systems (amplifiers, filters) used at each end of the link.

(Bandwidth is akin to the number of lanes in a highway; a wider highway can handle more cars per day [bits per second]. Of course, the highway becomes more expensive to construct and maintain in this case.)

Some physical links have larger usable bandwidth then others; for example, optical fiber has a much wider bandwidth then twisted-pair telephone wire.

To understand the relationship of bandwidth to pulse transmission rate and hence to bit rate, we now make an apparent digression into a brief study of Fourier Series and Linear Systems.

2. The Fourier Series for Periodic Signals

Through the idea of **Fourier Series**, we can represent **any periodic function of time as a linear combination of sines (and cosines)** of different frequencies. More generally, **any pulse waveform can be represented using sine and cosine time functions only.** This will provide the connection between link bandwidth and the practical limit on pulse widths that can propagate on the link.

(Note: $\sin(2\pi f_0 t + \phi)$ is a sinusoidal function of time (t) with a **frequency** f_0 Hz . Its value at t=0 is $\sin(\phi)$, where ϕ is the **phase** of the sinusoid. This sinewave makes one complete oscillation every $T=1/f_0$ seconds; T is called the **period** of the sinewave. The Fourier series allows arbitrary periodic functions of time to be expressed in terms of sine and cosine time functions.)

2.1 Fourier Series Representation:

For **any periodic function** x(t) with repeating period T and repetition frequency $f_0=1/T$, the ***Fourier series*** is an expansion of the function in terms of sine and cosine functions as follows:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t) + \sum_{n=1}^{\infty} b_n \sin(2\pi n f_0 t) \qquad (1)$$

Here the a_n and b_n coefficients are obtained as the following integrals:

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt, \qquad \text{average, or "dc" value}$$

and

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(2\pi n f_0 t)\,dt, \qquad b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(2\pi n f_0 t)\,dt$$

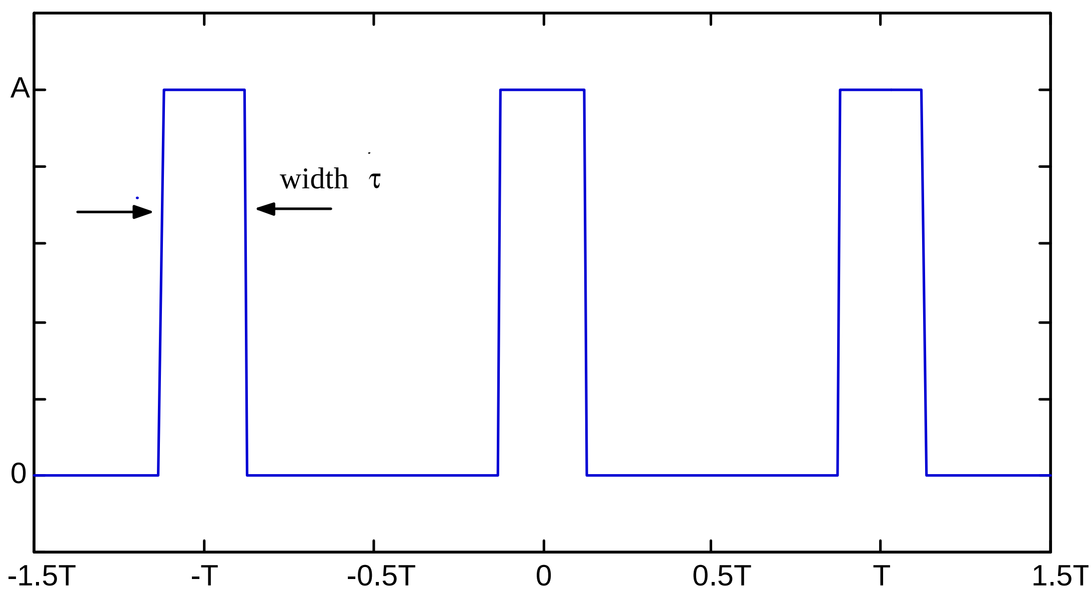
This representation is always possible for a periodic function (subject to some mild mathematical regularity conditions that are satisfied in all our applications.)

We do not give a proof of this representation here. However, many simple examples can be generated that illustrate how the right-hand side of Eq.(1) approximates better and better the function x(t) as more and more cosine and sine components are added. For a very nice "live" interactive demonstration of this representation, refer to the following:

http://www2.ece.jhu.edu/wjr/index.html

2.2 Example – Periodic Rectangular Pulse Train:

Consider a periodic pulse train v(t) of fundamental period T and amplitude A, each pulse in the train having a duration of t. Note that the fundamental frequency of the repeating pulse pattern is $f_0 = \frac{1}{T}$.



Periodic Pulse Train v(t)

For the pulse train v(t), we find easily that $a_0 = \frac{At}{T}$ and

$$a_n = \frac{2}{T} \int_{-t/2}^{t/2} A \cos(2\pi n f_0 t)\, dt$$

$$= \frac{2A}{T} \frac{\sin(2\pi n f_0 t)}{2\pi n f_0} \Big|_{-t/2}^{t/2} = \frac{A}{\pi n f_0 T} [\sin(\pi n f_0 t) - \sin(-\pi n f_0 t)]$$

$$= \frac{2A}{n\pi} \sin(\pi n f_0 t) \qquad (2)$$

The coefficients b_n are all 0 here because we end up with an integral of $A\sin(2\pi n f_0 t)$ over the interval $(-t/2,\ t/2)$ and since the sine function is an odd function this integrates to 0.

Therefore the Fourier series representation of this pulse train is

$$v(t) = \frac{At}{T} + \sum_{n=1}^{\infty} \frac{2A}{np} \sin(n\pi f_0 t) \cos(2\pi n f_0 t)$$

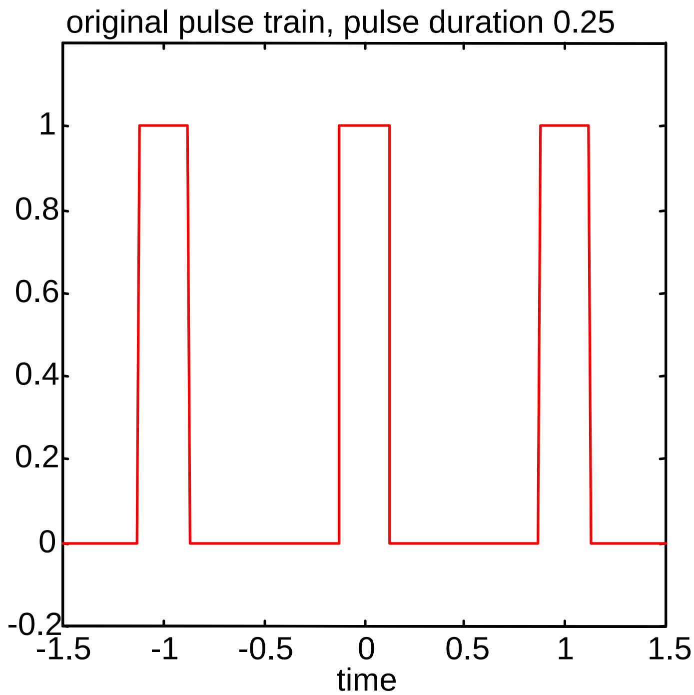
$$= a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t) \qquad (3)$$

where a_0= dc value, and $f_0 = \frac{1}{T}$

(Note that coefficients a_0 and $a_n = \frac{2A}{np} \sin(\pi n f_0 \tau)$ are *constants* for each n, not functions of time.)

Fourier Series Plots

Consider the periodic rectangular pulse train v(t) with period T=1 and amplitude A=1. Let the duration of each pulse within each period be $t = 0.25$.



Rectangular Pulse Train, period 1, amplitude 1

Figure 1 shows the reconstruction of the pulse train using the a_0 + first 9 coefficients a_n of the Fourier series (that is, using up to the term a_9 in the series of Eq. (3)).

Notice that the approximation is quite good, although there are some clear overshoots and undershoots.

Figure 2 shows the a_0 + 3-coefficient representation of the pulse train. In this case the maximum frequency used is 3 Hz (since f_0, the fundamental frequency, is 1 Hz). Note that the a_0 + four-coefficient representation would be exactly the same, since the a_4 coefficient is 0.

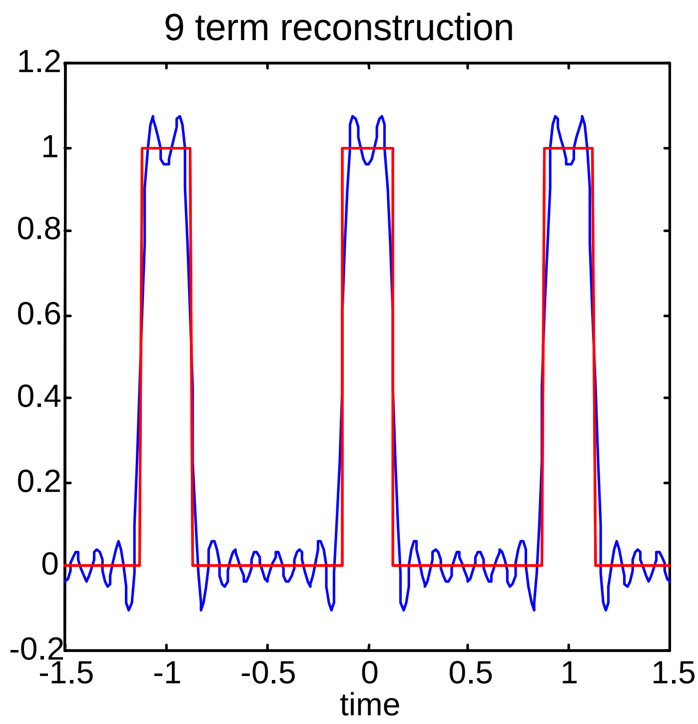


Figure 1

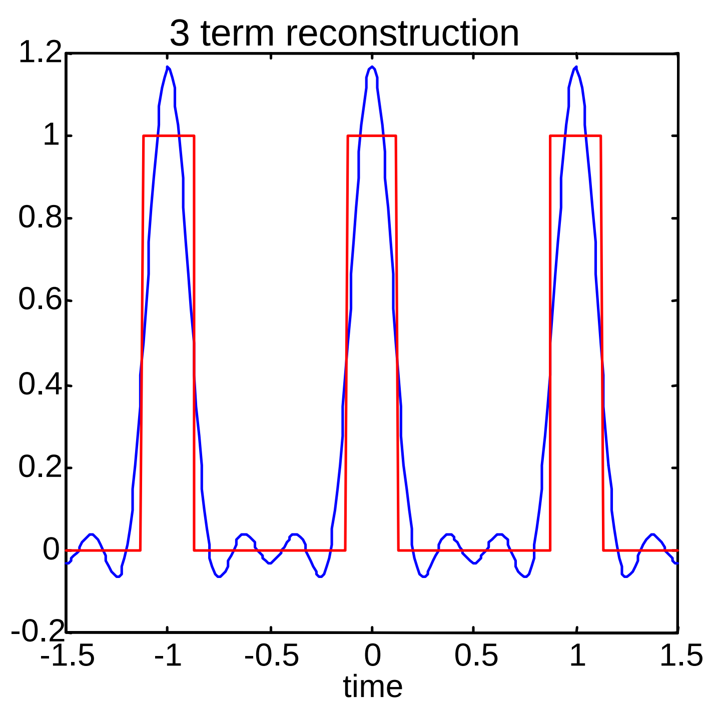


Figure 2