MATRICES & LINEAR EQUATIONS

SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS

We are used to solving two simultaneous linear equations in two unknowns by the methods of substitution and elimination. We will now look at methods of solution which are appropriate for more than two equations and more than two unknowns.

GUASSIAN ELIMINATION

Gaussian elimination is an extension of the methods used to solve two simultaneous linear equations in two unknowns. It eliminates one unknown at a time until the original system of equations is rearranged to an *echelon* form. Substitution is then used.

The number of equations need not be the same as the number of unknowns.

Each time a variable is eliminated, the number of equations is effectively reduced by 1.

The order of the equations can be changed at any time in order to make the calculations easier.

Shown opposite is the rearrangement of the equations:

$$w + 2x + y + z = 4 \quad \ldots(1)$$
$$w + x + 2y - z = 9 \quad \ldots(2)$$
$$-2w - x + y + z = 1 \quad \ldots(3)$$
$$-w + x - 3y + 2z = {}^{-}10 \quad \ldots(4)$$

to the echelon form:

$$w + 2x + y + z = 4 \quad \ldots(1)$$
$$x - y + 2z = -5 \quad \ldots(8)$$
$$y - \tfrac{1}{2}z = 4 \quad \ldots(12)$$
$$z = {}^{-}2 \quad \ldots(14)$$

Substitution gives the solution:

$$w = {}^{-}1,\ x = 2,\ y = 3,\ z = {}^{-}2$$

$$w + 2x + y + z = 4 \quad \ldots(1)$$
$$w + x + 2y - z = 9 \quad \ldots(2)$$
$$-2w - x + y + z = 1 \quad \ldots(3)$$
$$-w + x - 3y + 2z = {}^{-}10 \quad \ldots(4)$$

eliminate w:

$$(2) - (1): \quad -x + y - 2z = 5 \quad \ldots(5)$$
$$(3) + 2\times(1): \quad 3x + 3y + 3z = 9 \quad \ldots(6)$$
$$(4) + (1): \quad 3x - 2y + 3z = {}^{-}6 \quad \ldots(7)$$

change coefficient of x to 1 and simplify (6):

$$-1\times(5): \quad x - y + 2z = {}^{-}5 \quad \ldots(8)$$
$$\tfrac{1}{3}\times(6): \quad x + y + z = 3 \quad \ldots(9)$$
$$3x - 2y + 3z = {}^{-}6 \quad \ldots(7)$$

eliminate x:

$$(9) - (8): \quad 2y - z = 8 \quad \ldots(10)$$
$$(7) - 3\times(8): \quad y - 3z = 9 \quad \ldots(11)$$

change coefficient of y to 1:

$$\tfrac{1}{2}\times(10): \quad y - \tfrac{1}{2}z = 4 \quad \ldots(12)$$
$$y - 3z = 9 \quad \ldots(11)$$

eliminate y:

$$(11) - (12): \quad -\tfrac{5}{2}z = 5 \quad \ldots(13)$$

change coefficient of z to 1:

$$-\tfrac{2}{5}\times(13): \quad z = {}^{-}2 \quad \ldots(14)$$

substituting $z = {}^{-}2$ in (12):

$$y - \tfrac{1}{2}\times{}^{-}2 = 4$$
$$y = 3$$

substituting $z = {}^{-}2$ and $y = 3$ in (8):

$$x - 3 + 2\times{}^{-}2 = {}^{-}5$$
$$x = 2$$

substituting $z = {}^{-}2$, $y = 3$ and $x = 2$ in (1):

$$w + 2\times 2 + 3 - 2 = 4$$
$$w = {}^{-}1$$

$$\therefore\ w = {}^{-}1,\ x = 2,\ y = 3,\ z = {}^{-}2$$

CONSISTENT & INCONSISTENT LINEAR EQUATIONS

Equations which have no solution are called *inconsistent*. Guassian elimination leads to results such as $y = 4$ and $y = {}^{-}3$.

Equations which can be solved are called *consistent*. Consistent equations may give rise to:

- a unique solution eg. $w = {}^{-}1,\ x = 2,\ y = 3,\ z = {}^{-}2$
- an infinite number of solutions eg. $w = {}^{-}1,\ x = 2,\ y = z + 5$

DEPENDENT & INDEPENDENT LINEAR EQUATIONS

If Guassian elimination results in a smaller number of equations than the original number, then the equations are called *dependent*. Equations are dependent if at least one equation can be written as a combination of others. Eg. equation (3) below can be obtained as equation (2) minus equation (1).

$$\begin{aligned} w - x + 2y - z &= 4 && \ldots(1) \\ 2w + 3x - y + z &= 2 && \ldots(2) \\ w + 4x - 3y + 2z &= {}^{-}2 && \ldots(3) \\ 3w + 2x - 4y + z &= 10 && \ldots(4) \end{aligned}$$

In solving this system of equations, one of equations (1), (2) or (3) can be ignored.

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☺ A linear equation in 2 unknowns represents a straight line. Solution of simultaneous linear equations in 2 unknowns is equivalent to finding points of intersection of straight lines. Give examples of 2 equations in 2 unknowns which result in no solutions, a unique solution and an infinite number of solutions.

☺ A linear equation in 3 unknowns represents a plane in 3 dimensional space. Solution of simultaneous linear equations in 3 unknowns is equivalent to finding points of intersection of planes. What would be true about the corresponding planes if 3 equations in 3 unknowns result in no solutions, a unique solution and an infinite number of solutions?

MATRIX NOTATION

The equations:

$$\begin{aligned} w + 2x + y + z &= 4 && \ldots(1) \\ w + x + 2y - z &= 9 && \ldots(2) \\ -2w - x + y + z &= 1 && \ldots(3) \\ -w + x - 3y + 2z &= {}^{-}10 && \ldots(4) \end{aligned}$$

can be written as **A X** = **V** where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & {}^{-}1 \\ {}^{-}2 & {}^{-}1 & 1 & 1 \\ {}^{-}1 & 1 & {}^{-}3 & 2 \end{pmatrix},\quad \mathbf{X} = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \mathbf{V} = \begin{pmatrix} 4 \\ 9 \\ 1 \\ {}^{-}10 \end{pmatrix}$$

A is the *coefficient matrix*, **X** is the *variable vector* and **V** is the *value vector*.

The coefficient matrix and the value vector can be combined to produce the *augmented matrix*:

$$(\mathbf{A}\,|\,\mathbf{V}) = \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 4 \\ 1 & 1 & 2 & ^-1 & 9 \\ ^-2 & ^-1 & 1 & 1 & 1 \\ ^-1 & 1 & ^-3 & 2 & ^-10 \end{array}\right)$$

GUASSIAN ELIMINATION USING THE AUGMENTED MATRIX

Guassian elimination can be performed as operations on the rows of the augmented matrix. The order of the rows can be changed at any time in order to make the calculations easier. The final form of the augmented matrix is an echelon matrix.

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 4 \\ 1 & 1 & 2 & ^-1 & 9 \\ ^-2 & ^-1 & 1 & 1 & 1 \\ ^-1 & 1 & ^-3 & 2 & ^-10 \end{array}\right) \to \begin{array}{c} \\ R_2 - R_1 \\ R_3 + 2\times R_1 \\ R_4 + R_1 \end{array} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 4 \\ 0 & ^-1 & 1 & ^-2 & 5 \\ 0 & 3 & 3 & 3 & 9 \\ 0 & 3 & ^-2 & 3 & ^-6 \end{array}\right) \to$$

$$\begin{array}{c} \\ ^-1\times R_2 \\ \frac{1}{3}\times R_3 \\ \\ \end{array} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 4 \\ 0 & 1 & ^-1 & 2 & ^-5 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 3 & ^-2 & 3 & ^-6 \end{array}\right) \to \begin{array}{c} \\ \\ R_3 - R_2 \\ R_4 - 3\times R_2 \end{array} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 4 \\ 0 & 1 & ^-1 & 2 & ^-5 \\ 0 & 0 & 2 & ^-1 & 8 \\ 0 & 0 & 1 & ^-3 & 9 \end{array}\right) \to$$

$$\begin{array}{c} \\ \\ \frac{1}{2}\times R_3 \\ \\ \end{array} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 4 \\ 0 & 1 & ^-1 & 2 & ^-5 \\ 0 & 0 & 1 & -\frac{1}{2} & 4 \\ 0 & 0 & 1 & ^-3 & 9 \end{array}\right) \to \begin{array}{c} \\ \\ \\ R_4 - R_3 \end{array} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 4 \\ 0 & 1 & ^-1 & 2 & ^-5 \\ 0 & 0 & 1 & -\frac{1}{2} & 4 \\ 0 & 0 & 0 & -\frac{5}{2} & 5 \end{array}\right) \to$$

$$\begin{array}{c} \\ \\ \\ -\frac{2}{5}\times R_4 \end{array} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 4 \\ 0 & 1 & ^-1 & 2 & ^-5 \\ 0 & 0 & 1 & -\frac{1}{2} & 4 \\ 0 & 0 & 0 & 1 & ^-2 \end{array}\right) \to \begin{array}{r} w + 2x + y + z = 4 \\ x - y + 2z = -5 \\ y - \frac{1}{2}z = 4 \\ z = {}^-2 \end{array} \quad \text{etc.}$$

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GUASS-JORDAN METHOD USING THE AUGMENTED MATRIX

In this variation of Guassian elimination, 0's are produced above as well as below the 1's. The order of the rows can be changed at any time in order to make the calculations easier. More row operations are needed than in Guassian elimination.

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 4 \\ 1 & 1 & 2 & ^-1 & 9 \\ ^-2 & ^-1 & 1 & 1 & 1 \\ ^-1 & 1 & ^-3 & 2 & ^-10 \end{array}\right) \to \begin{array}{c} \\ R_2 - R_1 \\ R_3 + 2\times R_1 \\ R_4 + R_1 \end{array} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 4 \\ 0 & ^-1 & 1 & ^-2 & 5 \\ 0 & 3 & 3 & 3 & 9 \\ 0 & 3 & ^-2 & 3 & ^-6 \end{array}\right) \to$$

$$\begin{array}{l} \\ {}^{-}1\times R_2 \\ \tfrac{1}{3}\times R_3 \\ \\ \end{array} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 4 \\ 0 & 1 & {}^{-}1 & 2 & {}^{-}5 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 3 & {}^{-}2 & 3 & {}^{-}6 \end{array}\right) \rightarrow \begin{array}{l} R_1 - 2\times R_2 \\ \\ R_3 - R_2 \\ R_4 - 3\times R_2 \end{array} \left(\begin{array}{cccc|c} 1 & 0 & 3 & {}^{-}3 & 14 \\ 0 & 1 & {}^{-}1 & 2 & {}^{-}5 \\ 0 & 0 & 2 & {}^{-}1 & 8 \\ 0 & 0 & 1 & {}^{-}3 & 9 \end{array}\right) \rightarrow$$

$$\begin{array}{l} \\ \\ \tfrac{1}{2}\times R_3 \\ \\ \end{array} \left(\begin{array}{cccc|c} 1 & 0 & 3 & {}^{-}3 & 14 \\ 0 & 1 & {}^{-}1 & 2 & {}^{-}5 \\ 0 & 0 & 1 & -\tfrac{1}{2} & 4 \\ 0 & 0 & 1 & {}^{-}3 & 9 \end{array}\right) \rightarrow \begin{array}{l} R_1 - 3\times R_3 \\ R_2 + R_3 \\ \\ R_4 - R_3 \end{array} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -\tfrac{3}{2} & 2 \\ 0 & 1 & 0 & \tfrac{3}{2} & {}^{-}1 \\ 0 & 0 & 1 & -\tfrac{1}{2} & 4 \\ 0 & 0 & 0 & -\tfrac{5}{2} & 5 \end{array}\right) \rightarrow$$

$$\begin{array}{l} \\ \\ \\ -\tfrac{2}{5}\times R_4 \end{array} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -\tfrac{3}{2} & 2 \\ 0 & 1 & 0 & \tfrac{3}{2} & {}^{-}1 \\ 0 & 0 & 1 & -\tfrac{1}{2} & 4 \\ 0 & 0 & 0 & 1 & {}^{-}2 \end{array}\right) \rightarrow \begin{array}{l} R_1 + \tfrac{3}{2}\times R_4 \\ R_2 - \tfrac{3}{2}\times R_4 \\ R_3 + \tfrac{1}{2}\times R_4 \\ \\ \end{array} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & {}^{-}1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & {}^{-}2 \end{array}\right)$$

The last matrix gives the solution $w = {}^{-}1,\ x = 2,\ y = 3,\ z = {}^{-}2$

HOMOGENEOUS EQUATIONS

A *homogeneous* system of linear equations has the constant equal to zero in each equation. The zero vector is always a solution of homogeneous linear equations and is sometimes called the *trivial solution*. The problem is therefore to find any other solutions.

Eg. The system of equations $x + y + z = 0,\ x + 2y + 3z = 0$ rearranges to $x = z,\ y = {}^{-}2z$ which represents an infinite number of solutions.

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