MATRICES & LINEAR EQUATIONS

INVERSE MATRIX

Square matrices which have an inverse are called *non-singular* or *invertible*. The inverse of matrix **A** is written as $\mathbf{A}^{-1}$ (not $\frac{1}{\mathbf{A}}$). Non-singular matrices of a certain size form a group under multiplication.

If $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & ^{-}1 \\ ^{-}2 & ^{-}1 & 1 & 1 \\ ^{-}1 & 1 & ^{-}3 & 2 \end{pmatrix}$ has inverse $\mathbf{A}^{-1}$, then $\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}_4$

FINDING THE INVERSE MATRIX BY ROW OPERATIONS

If $\mathbf{A}^{-1} = \begin{pmatrix} a & e & i & m \\ b & f & j & n \\ c & g & k & p \\ d & h & l & q \end{pmatrix}$, then $\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & ^{-}1 \\ ^{-}2 & ^{-}1 & 1 & 1 \\ ^{-}1 & 1 & ^{-}3 & 2 \end{pmatrix}\begin{pmatrix} a & e & i & m \\ b & f & j & n \\ c & g & k & p \\ d & h & l & q \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

This is equivalent to solving:

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & ^{-}1 \\ ^{-}2 & ^{-}1 & 1 & 1 \\ ^{-}1 & 1 & ^{-}3 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & ^{-}1 \\ ^{-}2 & ^{-}1 & 1 & 1 \\ ^{-}1 & 1 & ^{-}3 & 2 \end{pmatrix}\begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & ^{-}1 \\ ^{-}2 & ^{-}1 & 1 & 1 \\ ^{-}1 & 1 & ^{-}3 & 2 \end{pmatrix}\begin{pmatrix} i \\ j \\ k \\ l \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & ^{-}1 \\ ^{-}2 & ^{-}1 & 1 & 1 \\ ^{-}1 & 1 & ^{-}3 & 2 \end{pmatrix}\begin{pmatrix} m \\ n \\ p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Each of these can be solved individually by the Guass-Jordan method or solved simultaneously by starting with the augmented matrix:

$$\left(\mathbf{A} \mid \mathbf{I}_4\right) = \left(\begin{array}{cccc|cccc} 1 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & ^{-}1 & 0 & 1 & 0 & 0 \\ ^{-}2 & ^{-}1 & 1 & 1 & 0 & 0 & 1 & 0 \\ ^{-}1 & 1 & ^{-}3 & 2 & 0 & 0 & 0 & 1 \end{array}\right)$$

and using row operations (including row swaps) to produce the augmented matrix $\left(\mathbf{I}_4 \mid \mathbf{A}^{-1}\right)$. If row operations give a row of 0's, then the matrix is singular and has no inverse.

The calculations below give $\mathbf{A}^{-1} = \begin{pmatrix} \frac{3}{5} & ^{-}1 & -\frac{2}{5} & -\frac{3}{5} \\ -\frac{4}{15} & 1 & \frac{1}{15} & \frac{3}{5} \\ \frac{1}{5} & 0 & \frac{1}{5} & -\frac{1}{5} \\ \frac{11}{15} & ^{-}1 & \frac{1}{15} & -\frac{2}{5} \end{pmatrix}$.

$$\left(\begin{array}{cccc|cccc} 1 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & -1 & 0 & 1 & 0 & 0 \\ -2 & -1 & 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & -3 & 2 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{array}{c} \\ R_2 - R_1 \\ R_3 + 2\times R_1 \\ R_4 + R_1 \end{array} \quad \left(\begin{array}{cccc|cccc} 1 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 3 & -2 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{array}{c} \\ -1\times R_2 \\ \\ \\ \end{array} \quad \left(\begin{array}{cccc|cccc} 1 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 & 1 & -1 & 0 & 0 \\ 0 & 3 & 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 3 & -2 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{array}{c} R_1 - 2\times R_2 \\ \\ R_3 - 3\times R_2 \\ R_4 - 3\times R_2 \end{array} \quad \left(\begin{array}{cccc|cccc} 1 & 0 & 3 & -3 & -1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 2 & 1 & -1 & 0 & 0 \\ 0 & 0 & 6 & -3 & -1 & 3 & 1 & 0 \\ 0 & 0 & 1 & -3 & -2 & 3 & 0 & 1 \end{array}\right)$$

$$\begin{array}{c} \\ \\ \frac{1}{6}\times R_3 \\ \\ \end{array} \quad \left(\begin{array}{cccc|cccc} 1 & 0 & 3 & -3 & -1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 2 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{6} & \frac{1}{2} & \frac{1}{6} & 0 \\ 0 & 0 & 1 & -3 & -2 & 3 & 0 & 1 \end{array}\right)$$

$$\begin{array}{c} R_1 - 3\times R_3 \\ R_2 + R_3 \\ \\ R_4 - R_3 \end{array} \quad \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{3}{2} & \frac{5}{6} & -\frac{1}{2} & \frac{1}{6} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{6} & \frac{1}{2} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & -\frac{5}{2} & -\frac{11}{6} & \frac{5}{2} & -\frac{1}{6} & 1 \end{array}\right)$$

$$\begin{array}{c} \\ \\ \\ -\frac{2}{5}\times R_4 \end{array} \quad \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{3}{2} & \frac{5}{6} & -\frac{1}{2} & \frac{1}{6} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{6} & \frac{1}{2} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 1 & \frac{11}{15} & -1 & \frac{1}{15} & -\frac{2}{5} \end{array}\right)$$

$$\begin{array}{c} R_1 + \frac{3}{2}\times R_4 \\ R_2 - \frac{3}{2}\times R_4 \\ R_3 + \frac{1}{2}\times R_4 \\ \\ \end{array} \quad \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{3}{5} & -1 & -\frac{2}{5} & -\frac{3}{5} \\ 0 & 1 & 0 & 0 & -\frac{4}{15} & 1 & \frac{1}{15} & \frac{3}{5} \\ 0 & 0 & 1 & 0 & \frac{1}{5} & 0 & \frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & 0 & 1 & \frac{11}{15} & -1 & \frac{1}{15} & -\frac{2}{5} \end{array}\right)$$

SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS USING THE INVERSE MATRIX

If a system of n linear equations in n unknowns has a unique solution, then the inverse matrix can be used to give that solution. Both sides of the matrix equation are pre-multiplied by the inverse matrix (why not post-multiplied?).

$$\begin{aligned} \mathbf{A}\,\mathbf{X} &= \mathbf{V} \\ \mathbf{A}^{-1}\left(\mathbf{A}\,\mathbf{X}\right) &= \mathbf{A}^{-1}\,\mathbf{V} \\ \left(\mathbf{A}^{-1}\,\mathbf{A}\right)\mathbf{X} &= \mathbf{A}^{-1}\,\mathbf{V} \\ \mathbf{I}\mathbf{X} &= \mathbf{A}^{-1}\,\mathbf{V} \\ \mathbf{X} &= \mathbf{A}^{-1}\,\mathbf{V} \end{aligned}$$

$$\begin{aligned} w+2x+y+z &= 4 \\ w+x+2y-z &= 9 \\ -2w-x+y+z &= 1 \\ -w+x-3y+2z &= {}^{-}10 \end{aligned}$$

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & {}^{-}1 \\ {}^{-}2 & {}^{-}1 & 1 & 1 \\ {}^{-}1 & 1 & {}^{-}3 & 2 \end{pmatrix}\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \\ 1 \\ {}^{-}10 \end{pmatrix}$$

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & {}^{-}1 \\ {}^{-}2 & {}^{-}1 & 1 & 1 \\ {}^{-}1 & 1 & {}^{-}3 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 4 \\ 9 \\ 1 \\ {}^{-}10 \end{pmatrix}$$

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & {}^{-}1 & -\frac{2}{5} & -\frac{3}{5} \\ -\frac{4}{15} & 1 & \frac{1}{15} & \frac{3}{5} \\ \frac{1}{5} & 0 & \frac{1}{5} & -\frac{1}{5} \\ \frac{11}{15} & {}^{-}1 & \frac{1}{15} & -\frac{2}{5} \end{pmatrix}\begin{pmatrix} 4 \\ 9 \\ 1 \\ {}^{-}10 \end{pmatrix}$$

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} {}^{-}1 \\ 2 \\ 3 \\ {}^{-}2 \end{pmatrix}$$

$$w = {}^{-}1,\ x = 2,\ y = 3,\ z = {}^{-}2$$

☺ Ex 5.7 p 169, Ex 5.8 p177

☺ If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, show that $\mathbf{A}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

☺ Show that the following properties of inverse matrices hold for 2 x 2 matrices.

$$\left(\mathbf{A}\mathbf{B}\right)^{-1} = \mathbf{B}^{-1}\,\mathbf{A}^{-1} \qquad \left(k\,\mathbf{A}\right)^{-1} = k^{-1}\,\mathbf{A}^{-1} \qquad \left(\mathbf{A}^{\mathsf{T}}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{\mathsf{T}}$$

DETERMINANTS

Every *square matrix* has a number associated with it called the *determinant* of the matrix. The determinant of **A** is written as $|\mathbf{A}|$, $\det \mathbf{A}$ or Δ.

Determinant of a 1 x 1 matrix

If $\mathbf{A} = (a)$, then $|\mathbf{A}| = a$.

Determinant of a 2 x 2 matrix

If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $|\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a\,d - bc$.

Determinant of a 3 x 3 matrix

If $\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$,

then $|\mathbf{A}| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$.

Determinant of any size matrix

The determinant of a $n \times n$ matrix is found in terms of determinants of $(n-1) \times (n-1)$ matrices.

For any element of the matrix:

- the *minor* is the determinant of the sub-matrix obtained by deleting the row and column containing that element
- the *cofactor* is the minor with a + or - attached according to the alternating sign rule shown opposite.

$$\begin{vmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

Eg. Consider element b in $\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

Deleting the row and column containing b gives $\begin{pmatrix} \# & \# & \# \\ d & \# & f \\ g & \# & i \end{pmatrix}$.

The minor of b is $\begin{vmatrix} d & f \\ g & i \end{vmatrix}$ and the cofactor of b is $-\begin{vmatrix} d & f \\ g & i \end{vmatrix}$.

To find the determinant of a square matrix:

- choose any row (or column)
- for each element in the row (or column), calculate the product of the element and its cofactor
- sum the products for the row (or column).

PROPERTIES OF DETERMINANTS

1. $|\mathbf{I}| = 1$
2. $|\mathbf{A}\,\mathbf{B}| = |\mathbf{A}|\,|\mathbf{B}|$
3. $|\mathbf{A}^{\mathrm{T}}| = |\mathbf{A}|$
4. If a non-zero multiple of a row (or column) is added to another row (or column), then the determinant is unchanged.
5. If a row (or column) is the same as another row (or column), then the determinant is zero.
6. If two rows (or columns) are swapped, then the determinant changes sign.
7. If a row (or column) is multiplied by a constant, then the determinant is multiplied by the same constant.
8. If a row (or column) consists of all zeros, then the determinant is zero.
9. If all the elements below the leading diagonal are zeros, then the determinant is the product of the elements on the diagonal.

CALCULATION OF DETERMINANTS

Expanding on the first row:

$$\begin{vmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & {}^{-}1 \\ {}^{-}2 & {}^{-}1 & 1 & 1 \\ {}^{-}1 & 1 & {}^{-}3 & 2 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 2 & {}^{-}1 \\ {}^{-}1 & 1 & 1 \\ 1 & {}^{-}3 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 & {}^{-}1 \\ {}^{-}2 & 1 & 1 \\ {}^{-}1 & {}^{-}3 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 & {}^{-}1 \\ {}^{-}2 & {}^{-}1 & 1 \\ {}^{-}1 & 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 2 \\ {}^{-}2 & {}^{-}1 & 1 \\ {}^{-}1 & 1 & {}^{-}3 \end{vmatrix}$$

$$= \left[1\begin{vmatrix} 1 & 1 \\ {}^{-}3 & 2 \end{vmatrix} - 2\begin{vmatrix} {}^{-}1 & 1 \\ 1 & 2 \end{vmatrix} + {}^{-}1\begin{vmatrix} {}^{-}1 & 1 \\ 1 & {}^{-}3 \end{vmatrix}\right] - 2\left[1\begin{vmatrix} 1 & 1 \\ {}^{-}3 & 2 \end{vmatrix} - 2\begin{vmatrix} {}^{-}2 & 1 \\ {}^{-}1 & 2 \end{vmatrix} + {}^{-}1\begin{vmatrix} {}^{-}2 & 1 \\ {}^{-}1 & {}^{-}3 \end{vmatrix}\right]$$

$$+ \left[1\begin{vmatrix} {}^{-}1 & 1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} {}^{-}2 & 1 \\ {}^{-}1 & 2 \end{vmatrix} + {}^{-}1\begin{vmatrix} {}^{-}2 & {}^{-}1 \\ {}^{-}1 & 1 \end{vmatrix}\right] - \left[1\begin{vmatrix} {}^{-}1 & 1 \\ 1 & {}^{-}3 \end{vmatrix} - 1\begin{vmatrix} {}^{-}2 & 1 \\ {}^{-}1 & {}^{-}3 \end{vmatrix} + 2\begin{vmatrix} {}^{-}2 & {}^{-}1 \\ {}^{-}1 & 1 \end{vmatrix}\right]$$

$$= \left[\left(1\times 2 - 1\times {}^{-}3\right) - 2\left({}^{-}1\times 2 - 1\times 1\right) - \left({}^{-}1\times {}^{-}3 - 1\times 1\right)\right]$$

$$-2\left[\left(1\times 2 - 1\times {}^{-}3\right) - 2\left({}^{-}2\times 2 - 1\times {}^{-}1\right) - \left({}^{-}2\times {}^{-}3 - 1\times {}^{-}1\right)\right]$$

$$+\left[\left({}^{-}1\times 2 - 1\times 1\right) - \left({}^{-}2\times 2 - 1\times {}^{-}1\right) - \left({}^{-}2\times 1 - {}^{-}1\times {}^{-}1\right)\right]$$

$$-\left[\left({}^{-}1\times {}^{-}3 - 1\times 1\right) - \left({}^{-}2\times {}^{-}3 - 1\times {}^{-}1\right) + 2\left({}^{-}2\times 1 - {}^{-}1\times {}^{-}1\right)\right]$$

$$= (5+6-2) - 2(5+6-7) + \left({}^{-}3+3+3\right) - (2-7-6)$$

$$= 9 - 8 + 3 + 11$$

$$= 15$$

Shown below is a simpler calculation based on the properties of determinants. The determinant is expanded on the first column.

$$\begin{vmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & ^-1 \\ ^-2 & ^-1 & 1 & 1 \\ ^-1 & 1 & ^-3 & 2 \end{vmatrix}$$

$$= \begin{matrix} \\ R_2 - R_1 \\ R_3 + 2 \times R_1 \\ R_4 + R_1 \end{matrix} \begin{vmatrix} 1 & 2 & 1 & 1 \\ 0 & ^-1 & 1 & ^-2 \\ 0 & 3 & 3 & 3 \\ 0 & 3 & ^-2 & 3 \end{vmatrix}$$

$$= 1 \begin{vmatrix} ^-1 & 1 & ^-2 \\ 3 & 3 & 3 \\ 3 & ^-2 & 3 \end{vmatrix}$$

$$= ^-1 \begin{vmatrix} 3 & 3 \\ ^-2 & 3 \end{vmatrix} - 1 \begin{vmatrix} 3 & 3 \\ 3 & 3 \end{vmatrix} + ^-2 \begin{vmatrix} 3 & 3 \\ 3 & ^-2 \end{vmatrix}$$

$$= ^-1\left(3 \times 3 - 3 \times ^-2\right) - \left(3 \times 3 - 3 \times 3\right) - 2\left(3 \times ^-2 - 3 \times 3\right)$$

$$= ^-15 - 0 + 30$$

$$= 15$$

☺ Ex 5.9 p 181, Ex 5.10 p 184
☺ Show that the property $|\mathbf{A}\,\mathbf{B}| = |\mathbf{A}||\mathbf{B}|$ holds for 2 x 2 matrices.

<u>FINDING THE INVERSE MATRIX BY USING THE DETERMINANT</u>

If $|\mathbf{A}| \neq 0$, then $\mathbf{A}^{-1} = \dfrac{\text{adj}\,\mathbf{A}}{|\mathbf{A}|}$. If $|\mathbf{A}| = 0$, then **A** does not have an inverse.

The *adjoint matrix* of **A** or $\text{adj}\,\mathbf{A}$ is the transpose of the *cofactor matrix* of **A**. The cofactor matrix of A is the matrix obtained by replacing each element with its cofactor.

The inverse of $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & ^-1 \\ ^-2 & ^-1 & 1 & 1 \\ ^-1 & 1 & ^-3 & 2 \end{pmatrix}$ is found below.

cofactor matrix of **A**

$$
=\begin{pmatrix}
\begin{vmatrix}1&2&{}^{-}1\\{}^{-}1&1&1\\1&{}^{-}3&2\end{vmatrix} & -\begin{vmatrix}1&2&{}^{-}1\\{}^{-}2&1&1\\{}^{-}1&{}^{-}3&2\end{vmatrix} & \begin{vmatrix}1&1&{}^{-}1\\{}^{-}2&{}^{-}1&1\\{}^{-}1&1&2\end{vmatrix} & -\begin{vmatrix}1&1&2\\{}^{-}2&{}^{-}1&1\\{}^{-}1&1&{}^{-}3\end{vmatrix}\\
-\begin{vmatrix}2&1&1\\{}^{-}1&1&1\\1&{}^{-}3&2\end{vmatrix} & \begin{vmatrix}1&1&1\\{}^{-}2&1&1\\{}^{-}1&{}^{-}3&2\end{vmatrix} & -\begin{vmatrix}1&2&1\\{}^{-}2&{}^{-}1&1\\{}^{-}1&1&2\end{vmatrix} & \begin{vmatrix}1&2&1\\{}^{-}2&{}^{-}1&1\\{}^{-}1&1&{}^{-}3\end{vmatrix}\\
\begin{vmatrix}2&1&1\\1&2&{}^{-}1\\1&{}^{-}3&2\end{vmatrix} & -\begin{vmatrix}1&1&1\\1&2&{}^{-}1\\{}^{-}1&{}^{-}3&2\end{vmatrix} & \begin{vmatrix}1&2&1\\1&1&{}^{-}1\\{}^{-}1&1&2\end{vmatrix} & -\begin{vmatrix}1&2&1\\1&1&2\\{}^{-}1&1&{}^{-}3\end{vmatrix}\\
-\begin{vmatrix}2&1&1\\1&2&{}^{-}1\\{}^{-}1&1&1\end{vmatrix} & \begin{vmatrix}1&1&1\\1&2&{}^{-}1\\{}^{-}2&1&1\end{vmatrix} & -\begin{vmatrix}1&2&1\\1&1&{}^{-}1\\{}^{-}2&{}^{-}1&1\end{vmatrix} & \begin{vmatrix}1&2&1\\1&1&2\\{}^{-}2&{}^{-}1&1\end{vmatrix}
\end{pmatrix}
= \;\ldots = \begin{pmatrix}9&{}^{-}4&3&11\\{}^{-}15&15&0&{}^{-}15\\{}^{-}6&1&3&1\\{}^{-}9&9&{}^{-}3&{}^{-}6\end{pmatrix}
$$

$$
\text{adj}\,\mathbf{A} = \text{transpose of cofactor matrix of } \mathbf{A} = \begin{pmatrix}9&{}^{-}15&{}^{-}6&{}^{-}9\\{}^{-}4&15&1&9\\3&0&3&{}^{-}3\\11&{}^{-}15&1&{}^{-}6\end{pmatrix}
$$

$$
\mathbf{A}^{-1} = \frac{\text{adj}\,\mathbf{A}}{|\mathbf{A}|} = \frac{1}{15}\begin{pmatrix}9&{}^{-}15&{}^{-}6&{}^{-}9\\{}^{-}4&15&1&9\\3&0&3&{}^{-}3\\11&{}^{-}15&1&{}^{-}6\end{pmatrix} = \begin{pmatrix}\frac{3}{5}&{}^{-}1&-\frac{2}{5}&-\frac{3}{5}\\-\frac{4}{15}&1&\frac{1}{15}&\frac{3}{5}\\\frac{1}{5}&0&\frac{1}{5}&-\frac{1}{5}\\\frac{11}{15}&{}^{-}1&\frac{1}{15}&-\frac{2}{5}\end{pmatrix}
$$

<u>Inverse of a 2 x 2 matrix</u>

If $\mathbf{A} = \begin{pmatrix}a&b\\c&d\end{pmatrix}$: cofactor matrix of $\mathbf{A} = \begin{pmatrix}d&-c\\-b&a\end{pmatrix}$

$$
\text{adj}\,\mathbf{A} = \text{transpose of cofactor matrix of } \mathbf{A} = \begin{pmatrix}d&-b\\-c&a\end{pmatrix}
$$

$$
\mathbf{A}^{-1} = \frac{\text{adj}\,\mathbf{A}}{|\mathbf{A}|} = \frac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}
$$

<u>SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS</u>
<u>AND THE VALUE OF THE DETERMINANT</u>

A system of n linear equations in n unknowns $\mathbf{A}\mathbf{X} = \mathbf{V}$ has a unique solution ($\mathbf{X} = \mathbf{A}^{-1}\mathbf{V}$).

$\Leftrightarrow$ The inverse matrix $\mathbf{A}^{-1}$ exists.

$\Leftrightarrow$ $|\mathbf{A}| \neq 0$

HOMOGENEOUS LINEAR EQUATIONS AND THE VALUE OF THE DETERMINANT

Consider a homogeneous system of n linear equations in n unknowns $\mathbf{A}\mathbf{X} = \mathbf{0}$.

The trivial solution (all unknowns equal to zero) is unique. $\Leftrightarrow |\mathbf{A}| \neq 0$

If $|\mathbf{A}| = 0$, then the trivial solution is not unique ie. the equations have an infinite number of solutions.

SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS BY CRAMER'S RULE

If a system of n linear equations in n unknowns $\mathbf{A}\mathbf{X} = \mathbf{V}$ has a unique solution, then Cramer's rule gives that solution - if $\mathbf{A}_i$ is the matrix $\mathbf{A}$ with the i-th column replaced by $\mathbf{V}$, then the value of the i-th unknown is $\dfrac{|\mathbf{A}_i|}{|\mathbf{A}|}$.

Consider the example:

$$\begin{aligned} w + 2x + y + z &= 4 \\ w + x + 2y - z &= 9 \\ -2w - x + y + z &= 1 \\ -w + x - 3y + 2z &= {}^{-}10 \end{aligned}$$

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & {}^{-}1 \\ {}^{-}2 & {}^{-}1 & 1 & 1 \\ {}^{-}1 & 1 & {}^{-}3 & 2 \end{pmatrix}\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \\ 1 \\ {}^{-}10 \end{pmatrix}$$

$$w = \frac{\begin{vmatrix} 4 & 2 & 1 & 1 \\ 9 & 1 & 2 & {}^{-}1 \\ 1 & {}^{-}1 & 1 & 1 \\ {}^{-}10 & 1 & {}^{-}3 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & {}^{-}1 \\ {}^{-}2 & {}^{-}1 & 1 & 1 \\ {}^{-}1 & 1 & {}^{-}3 & 2 \end{vmatrix}} = \ldots = \frac{{}^{-}15}{15} = {}^{-}1, \quad x = \frac{\begin{vmatrix} 1 & 4 & 1 & 1 \\ 1 & 9 & 2 & {}^{-}1 \\ {}^{-}2 & 1 & 1 & 1 \\ {}^{-}1 & {}^{-}10 & {}^{-}3 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & {}^{-}1 \\ {}^{-}2 & {}^{-}1 & 1 & 1 \\ {}^{-}1 & 1 & {}^{-}3 & 2 \end{vmatrix}} = \ldots = \frac{30}{15} = 2, \quad \text{etc.}$$

☺ Ex 5.11 p 188, Ex 5.12 p 191

☺ Prove Cramer's rule for $ax + by = u$, $cx + dy = v$.

☺ Show that the area of the triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by the determinant opposite.

$$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

☺ Explain why the equation opposite is the equation of the line passing through the points (x_1, y_1) and (x_2, y_2).

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x \\ y_1 & y_2 & y \end{vmatrix} = 0$$