CHAPTER 17: EXPONENTIAL AND LOGARITHM FUNCTIONS

1. Exponential Functions

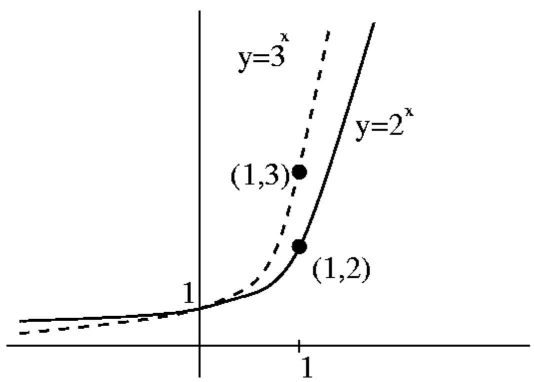
Definition 1.1. An *exponential function* is a function of the form

$$f(x) = A \cdot b^x \qquad (A,\ b \quad \text{constants})$$

(or $f(t) = Ab^t$ or $f(r) = Ab^r$ ...etc.) That is, it is a function with a variable *exponent*.

Example 1.1. The functions $f(x) = 2^x$, $g(x) = 3^x$, $h(x) = 100 \cdot 5^x$ are exponential functions.

Note the difference between the *exponential function* $f(x) = 2^x$ with fixed base and variable exponent or power, and the *power function* $f(x) = x^2$ with fixed exponent and variable base.



Exponential functions occur naturally:
- Radioactive decay
- unconstrained growth of a population

Example 1.2. The population of Mexico (1980-1983):

year	pop. (in millions)
1980	67.38
1981	69.13
1982	70.93
1983	72.77

The successsive ratios are (approximately) the same:

$$\frac{69.13}{67.38} \approx 1.026$$

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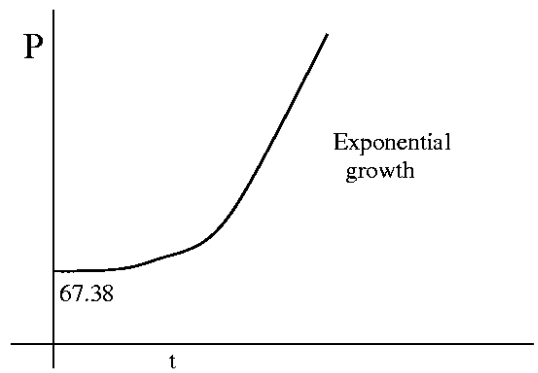
In each year, the population increases by a factor of 1.026.
Let $P(t)$ denote the population at time t (where t is the number of years since 1980):

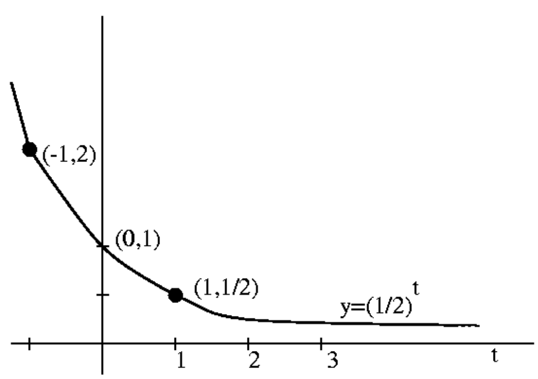
$$\begin{aligned} P(0) &= 67.38 = P_0 \\ P(1) &= 67.38 \cdot (1.026) \\ P(2) &= 67.38 \cdot (1.026) \cdot (1.026) = 67.38 \cdot (1.026)^2 \\ P(3) &= 67.38 \cdot (1.026)^3 \\ &\vdots \\ P(t) &= 67.38 \cdot (1.026)^t = P_0 b^t \end{aligned}$$

Thus the estimated population in 1990 is $P(10) = 67.38 \cdot (1.026)^{10} \approx 87.1$



Note: If $f(t) = Ab^t$, we get *exponential growth* if $b > 1$ and we get *exponential decay* if $b < 1$. (b is called the *base* of the exponential function.)

Example 1.3. Consider the exponential function $f(t) = (1/2)^t = 2^{-t}$:



This function exhibits exponential decay.

Example 1.4. For any given amount of radioactive potassium (${}^{44}_{19}\mathrm{K}$), the amount remaining one second later is 99.97%. Find a formula for the amount at time t. (Let A_0 be the amount at time 0).

Solution:

$$\begin{aligned} A(t) &= \text{amount at time } t \\ A(0) &= A_0 \\ A(1) &= A_0 \cdot (0.9997) \\ &\vdots \\ A(t) &= A_0 \cdot (0.9997)^t \end{aligned}$$

2. Half-Lives

Suppose that $A(t) = A_0 \cdot b^t$ with $b < 1$
(i.e., we have exponential decay).
Then there is a number $h > 0$ with

$$b^h = \frac{1}{2}$$

Thus, if t is any time, the amount at time $t + h$ is

$$\begin{aligned} A(t+h) &= A_0 \cdot b^{t+h} \\ &= A_0 \cdot b^t \cdot b^h \\ &= \frac{1}{2} A_0 b^t \\ &= \frac{1}{2} A(t) \\ &= \frac{1}{2} \text{ of the amount at time } t \end{aligned}$$

The number h is called the *half-life* or 1/2-*life* of the process.

Example 2.1. The 1/2-life of $^{44}_{19}\mathrm{K}$ is 22 mins.

Example 2.2. The 1/2-life of Carbon-14 is 5700 years.

A similar computation shows that if we have exponential growth

$$A(t) = A_0 b^t \qquad b > 1$$

then $b^d = 2$ for some $d > 0$.
This number d is called the *doubling time* for the process.

Example 2.3. The 1/2-life of carbon-14 is 5700 years. Find the formula for the amount at time t (t in years).
Solution: $A(t) = A_0 b^t$. What's b?
We know

$$A(5700) = A_0 b^{5700} = \frac{1}{2} A_0$$

So

$$b^{5700} = \frac{1}{2}$$

$$\implies b = \left(\frac{1}{2}\right)^{\frac{1}{5700}} = (0.5)^{0.000175} \approx 0.999878$$

So $A(t) \approx A_0 \cdot (0.999878)^t$.

Example 2.4. For a process of exponential decay

$$A(t) = A_0 b^t$$

determine the half-life.
Solution: We must solve

$$b^h = \frac{1}{2}$$

for h.

To do this, we need *logarithms*.

3. Logarithms

The logarithm function answers the question: *What power* of the number a is equal to the number c?

Definition 3.1. Suppose that $a > 0$ and $a^b = c$, then we say that b is *the logarithm of c to the base a* and write

$$\log_a(c) = b$$

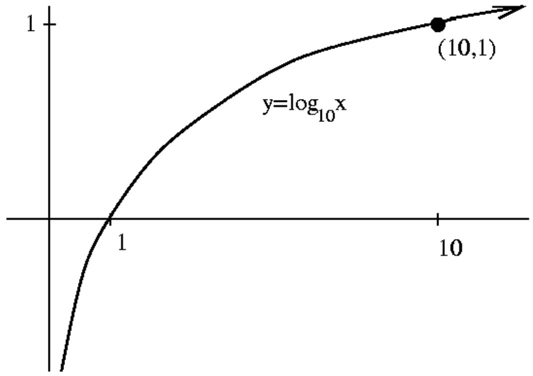
Example 3.1.

$$\begin{aligned} 2^3 = 8 &\implies 3 = \log_2 8 \\ 10^4 = 10000 &\implies 4 = \log_{10} 10000 \\ 10^{-1} = \frac{1}{10} &\implies -1 = \log_{10}\left(\frac{1}{10}\right) \\ a^0 = 1 &\implies 0 = \log_a(1) \end{aligned}$$

Thus '$\log_a c$' means the power that a must be raised to in order to get c.

Example 3.2. Thus if we ask: What power of 3 is equal to 47? we are asking for $\log_3(47)$. If we ask: what power of 10 is equal to 50?, we are asking for $\log_{10}(50)$.

Note that the 'input' in a logarithm must be a *positive number*; i.e., the *domain* of the function $f(x) = \log_a(x)$ is $(0, \infty)$.
Graph of $y = \log_{10} x$:



Notation: In elementary texts (and in this course), $\log_{10} x$ is simply denoted $\log x$, and is called the '*common logarithm*'.
Thus $\log x = y$ means $10^y = x$.

Example 3.3. So $\log(5) = 0.69897\ldots$ means $10^{0.69897\ldots} = 5$.

4. Properties of Logarithms

Theorem 4.1. *Fix $a > 0$.*

(1) $\log_a(1) = 0$
(2) $\log_a(a) = 1$
(3) $\log_a(c_1 \cdot c_2) = \log_a(c_1) + \log_a(c_2)$
(4) $\log_a\left(\frac{c_1}{c_2}\right) = \log_a(c_1) - \log_a(c_2)$
(5) $\log_a(c^d) = d\log_a(c)$

Proof:

(1) Since $a^0 = 1$.
(2) Since $a^1 = a$.
(3) Let $b_1 = \log_a(c_1)$ and $b_2 = \log_a(c_2)$. Then $a^{b_1} = c_1$ and $a^{b_2} = c_2$. So

$$a^{b_1+b_2} = c_1 \cdot c_2 \qquad \text{(First Law)}$$

$$\Longrightarrow \log_a(c_1c_2) = b_1 + b_2 = \log_a(c_1) + \log_a(c_2)$$

(4)

$$\log_a(c_1) = \log_a\left(\frac{c_1}{c_2} \cdot c_2\right) = \log_a\left(\frac{c_1}{c_2}\right) + \log_a(c_2)$$

(using 3.).
Now subtract $\log_a(c_2)$ from both sides.

(5) Let $b = \log_a(c)$. So $a^b = c$.
Thus $a^{bd} = (a^b)^d = c^d$ (Second Law).
So $\log_a(c^d) = bd = d\log_a(c)$

Note: Taking $d = -1$ in 5. gives

$$\log_a\left(\frac{1}{c}\right) = -\log_a(c)$$

Example 4.1. The formula for the amount of radioactive polonium is

$$A(t) = A_0(0.99506)^t \qquad (t \text{ in days })$$

What is the half-life?
Solution: We need to solve $.99506^h = 1/2$ for h:

$$\begin{aligned} \log(0.99506^h) &= \log(1/2) \\ h \cdot \log(0.99506) &= \log(1/2) \\ h &= \frac{\log(1/2)}{\log(0.99506)} \\ &\approx 140 \text{ (days)} \end{aligned}$$

Thus for a process of exponential decay we have,

$$\boxed{\text{half-life} = \frac{\log(1/2)}{\log(\text{base})}}$$

Similarly, for a process of exponential growth,

$$\boxed{\text{doubling time} = \frac{\log(2)}{\log(\text{base})}}$$

Example 4.2. Estimate the doubling time of the Mexican population.
Solution:

$$d = \frac{\log(2)}{\log(1.026)} = 27 \text{ (years)}$$

In general, logarithms are useful for solving equations where the unknown occurs as an exponent:

Example 4.3. Solve $7^x = 231 \cdot 5^x$ for x.
Solution: Take logs of both sides:

$$\begin{aligned} \log(7^x) &= \log(231 \cdot 5^x) \\ x\log(7) &= \log(231) + x\log(5) \\ x(\log(7) - \log(5)) &= \log(231) \\ x &= \frac{\log 231}{\log 7 - \log 5} \\ &\approx 16.175 \end{aligned}$$

5. Derivatives of logarithms and exponentials

What are

$$\frac{d}{dx}(2^x) \quad \text{or} \quad \frac{d}{dx}\log(x)?$$

We will not be in a position to answer these questions until we have defined the *natural logarithm* function, and to define the natural logarithm we have to first develop the *theory of integration*.