CHAPTER 25: DIFFERENTIAL EQUATIONS

1. Differential Equations and Mathematical Models

Differential equations are equations involving the derivative(s) of an unknown function $y = f(x)$ (or $x = x(t)$ or $A = A(t)$ or etc.).
Since a derivative is just a rate of change, such equations arise whenever we have information about the rate of change, growth or decay of some variable or quantity. For instance, many of the laws of Physics and Chemistry tell us about rates of change and thus are expressed as differential equations.

Example 1.1 (The Law of Exponential Growth)**.** The rate of growth at time t of a culture of bacteria is proportional to the amount present;i.e. the more bacteria present the greater the rate of production of new bacteria.
We can translate this into mathematics as follows:
Let $y = y(t) =$ amount of bacteria at time t. Then dy/dt is the rate of growth of the culture, and this is proportional to the amount at time t; i.e., dy/dt is proportional to y itself. Thus the law says:

$$\frac{dy}{dt} = Ky \qquad (K \text{ a constant})$$

The problem then is to solve this equation for y.

Example 1.2 (Newton's Law of Cooling)**.** This says:
The rate at which a body cools at a given time is proportional to the difference between the temperature of the body and the temperature of the environment (which we assume constant).
To translate this into mathematics: Let $T =$ temperature of the body at time t.
Let E denote the temperature of the environment (constant).
Then dT/dt is the rate of cooling and $T - E$ is the difference between the temperature of the body and the temperature of the environment. So

$$\frac{dT}{dt} = K(T - E) \qquad (K \text{ const.})$$

Example 1.3. Many differential equations can be derived from Newton's law of motion, which says that the acceleration of a body is proportional to the force acting on it. If $x = x(t)$ is the position of the body at time t, then the acceleration is $\frac{d^2x}{dt^2}$. Thus if we know the force F, then we get a differential equation of the form $\frac{d^2x}{dt^2} = a \cdot F$ for some constant a.
As a typical example, *Hooke's Law* says that the restoring force of a displaced spring is opposite in direction to the displacement and proportional

in size to the displacement. Thus, if x is the displacement (from $0 =$ rest) at time t, we get the differential equation for x:

$$\frac{d^2x}{dt^2} = -ax \ (a > 0 \text{ constant}).$$

2. Solving Differential Equations

Given a differential equation involving an unknown function y of the variable x, we would like to find explicit formulae for all possible solutions. This is not always possible (just as it is not always possible to solve numerical equations explicitly), but it is possible for certain special types of differential equations. We will consider some of the most elementary – but useful – types.

2.1. **Type 0: Antiderivatives.**

The simplest example of a differential equation is one where the derivative of the unknown function is explicitly given. Examples of such equations are:

$$\frac{dy}{dx} = x^3;$$
$$\frac{dx}{dt} = \sin(t);$$
$$\frac{dy}{dx} = \sin(x^3).$$

Observe the form of each equation. In each case the right-hand-side is a function of the *independent* variable x or t.

In each of these we have an unknown function (y or x) whose derivative is given and we want to determine all possibilities for y or x. This is just the problem of anti-differentiation or integration. Thus to find all solutions of these three differential equations, we just have to integrate the right-hand-side. The solutions are

$$y = \int x^3\,dx = x^4/4 + C$$
$$x = \int \sin(t)\,dt = -\cos(t) + C$$
$$y = \int \sin(x^3)\,dx = \int_a^x \sin(t^3)\,dt.$$

2.2. **Type I: Variables Separable.**

The general form of such a differential equation is:

$$\frac{dy}{dx} = f(x) \cdot g(y).$$

Thus the right-hand-side is the product of a function of the independent variable and a function of the dependent variable.

To solve such a differential equation:

Rewrite the equation, separating the variables:

$$\frac{1}{g(y)}dy = f(x)dx$$

Then integrate both sides (the left-hand-side with respect to y, the right hand side with respect to x).
[This can be justified more fully as follows:
First rewrite the equation as

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x).$$

Now integrate both sides with respect to x:

$$\int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int f(x)\,dx.$$

Finally, the rule of integration by substitution says

$$\int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int g(y)\,dy.]$$

Example 2.1.

$$\frac{dy}{dx} = y^2(x^3+1)$$

Find the *general solution* of this differential equation.
Solution:

$$\begin{aligned}
\frac{1}{y^2}dy &= (x^3+1)dx \\
\int \frac{1}{y^2}dy &= \int (x^3+1)dx \\
-\frac{1}{y} &= \frac{x^4}{4} + x + C \\
y &= \frac{-1}{(x^4/4)+x+C}
\end{aligned}$$

Example 2.2.

$$x\frac{dy}{dx} = e^{-y}$$

THis is separable:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{x}e^{-y} = \frac{1}{x}\frac{1}{e^y} \\
\int e^y\, dy &= \int \frac{1}{x}\, dx \\
e^y &= \ln|x| + C \\
y &= \ln(\ln|x| + C)
\end{aligned}$$

Example 2.3.

$$\frac{dy}{dt} = k \cdot y \qquad (k \text{ constant})$$

(This is the differential equation of a bacteria culture.)

$$\begin{aligned}
\int \frac{1}{y} dy &= \int k\ dt \\
\ln y &= kt + C \\
y &= e^{kt+C} \\
&= e^{kt}e^C \\
y &= Ae^{kt} \qquad (A = e^C \quad \text{const.} > 0)
\end{aligned}$$

Note: Suppose that we know the amount of bacteria at time $t = 0$: Let $y_0 = y(0)$ Thus $y_0 = Ae^0 = A$. Therefore the formula for the amount of bacteria at time t is

$$\boxed{y = y_0 e^{kt}}$$

This is the general solution. This computation gives a mathematical explanation of why bacteria in a Petri dish will grow according to an exponential law.

2.3. **Type II: 'First Order Linear'.** The meaning of the terms here:
'First Order' means that no second or higher derivatives occur.
'Linear' means that the equation is a linear function of y and its derivatives; there is no y^2, y^3, $\sin(y)$, $\left(\frac{dy}{dx}\right)^2$. ....
Thus a First Order Linear equation is one which can be written in th following *Standard Form*:

$$\frac{dy}{dx} + p(x)y = q(x) \qquad (*)$$

Method of Solution: First, introduce the *'integrating factor'*

$$f(x) = e^{\int p(x)dx}$$

So $f(x)$ satisfies $f'(x) = f(x)p(x)$.
Thus

$$\begin{aligned} \frac{d}{dx}(f(x)y) &= f(x)\frac{dy}{dx} + f'(x)y \\ &= f(x)\frac{dy}{dx} + f(x)p(x)y \\ &= f(x) \cdot \left[\frac{dy}{dx} + p(x)y\right] \\ &= f(x)q(x) \\ \text{So} \quad f(x)y &= \int f(x)q(x)dx \end{aligned}$$

To summarize: Thus to solve the differential equation $(*)$:

(1) Compute $\int p(x)dx = g(x)$
(2) Set $f(x) = e^{g(x)}$
(3) Compute $\int f(x)q(x)dx = H(x) + C$
(4) The general solution is

$$y = \frac{1}{f(x)}[H(x) + C]$$

Example 2.4.

$$\frac{dy}{dx} + \frac{2y}{x} = x^3$$

$$\frac{dy}{dx} + \underbrace{\frac{2}{x}}_{p(x)} y = \underbrace{x^3}_{q(x)}$$

Solution:

(1)

$$\begin{aligned} \int \frac{2}{x}dx &= 2\int \frac{1}{x}dx \\ &= 2\ln x \\ &= \ln(x^2) \end{aligned}$$

(2)

$$f(x) = e^{\ln(x^2)} = x^2$$

(3)

$$\int x^2 \cdot x^3 dx = \int x^5 dx = \frac{x^6}{6} + C$$

(4)

$$y = \frac{1}{x^2}\left(\frac{x^6}{6} + C\right)$$
$$y = \frac{x^4}{6} + \frac{C}{x^2}$$

Example 2.5. Find y satisfying

$$\frac{dy}{dx} + y = e^x$$

subject to the *initial condition* $y(0) = 1$.

Solution:

(1) $p(x) = 1$: $g(x) = \int 1\ dx = x$
(2) $f(x) = e^x$
(3)

$$\begin{aligned} H(x) &= \int e^x \cdot e^x\ dx \\ &= \int e^{2x} dx \\ &= \frac{1}{2}e^{2x} + C \end{aligned}$$

(4)

$$\begin{aligned} y &= \frac{1}{e^x}\left[\frac{1}{2}e^{2x} + C\right] \\ &= \frac{1}{2}e^x + \frac{C}{e^x} \\ &= \frac{1}{2}e^x + Ce^{-x} \end{aligned}$$

This gives the *general solution.*
Having found this, we use the *initial condition* to evaluate C:

$$\begin{aligned} 1 &= y(0) \\ &= \frac{1}{2}e^0 + Ce^0 = \frac{1}{2} + C \\ \Longrightarrow & C = 1/2 \end{aligned}$$

So

$$y = \frac{1}{2}e^x + \frac{1}{2}e^{-x} = \frac{e^x + e^{-x}}{2}$$

is the *particular solution* which satisfies the differential equation and the given initial condition.

Example 2.6. Solve

$$x\frac{dy}{dx} + y = x \sin x$$

Solution:

$$\frac{dy}{dx} + \underbrace{\frac{1}{x}}_{p(x)} y = \underbrace{\sin x}_{q(x)}$$

(1) $\int 1/x \ dx = \ln x$

(2)

$$f(x) = e^{lnx} = x$$

(3)

$$\int x \sin x \ dx = -x \cos x + \sin x + C$$

(4)

$$\begin{aligned} y &= \frac{1}{x}\left[-x\cos x + \sin x + C\right] \\ &= -\cos x + \frac{\sin x}{x} + \frac{C}{x} \end{aligned}$$

2.4. **Type III: '2nd order linear homogeneous'.** These are differential equations of the following form:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0 \quad (a, b \text{ constant})$$

Method of Solution:
Find the roots of

$$t^2 + at + b = 0 \quad \text{(the characteristic equation)}$$

Suupose that the roots are α and β.
If α, β are *real* and *distinct* then

$$\boxed{y = Ae^{\alpha x} + Be^{\beta x}}$$

is the general solution. (Here A and B stand for arbitrary independent constants.)

Example 2.7. Solve

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

Solution: The characteristic equation is $t^2 - 3t + 2 = 0$.
Roots: $1, 2$
So general solution is

$$y = Ae^x + Be^{2x}$$

Example 2.8. Find the particular solution of last example satisfying initial conditions

$$y(0) = 0,\ y'(0) = -1$$

Solution:

$$\begin{aligned} 0 = y(0) &= Ae^0 + Be^0 = A + B \\ y'(x) &= Ae^x + 2Be^{2x} \\ -1 = y'(0) &= A + 2B \\ \text{So } A = 1, &\qquad B = -1 \end{aligned}$$

$$y = e^x - e^{2x}$$

If the roots are equal to each other (i.e., $\alpha = \beta$), then the general solution is

$$\begin{aligned} y &= Ae^{\alpha x} + Bxe^{\alpha x} \\ &= (A + Bx)e^{\alpha x} \end{aligned}$$

Example 2.9. Find the general solution of

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$$

Solution:
$t^2 + 4t + 4 = 0$
Roots: $\alpha = \beta = -2$
General solution:

$$y = Ae^{-2x} + Bxe^{-2x}$$

2.5. **Type IV: Nonhomogeneous Equations.** These differ from the last type in that the right-hand-side is no longer 0, but is a function of x:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)$$

We'll consider only the special case where the right-hand-side is an exponential function

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = Ke^{cx} \quad (*)$$

where c is *not* a root of the characteristic equation.
Method of solution:
Suppose that y_p is any particular solution of $(*)$.
The general solution of $(*)$ is then

$$y_p + y_h$$

where $y_h =$ is the general solutionn of the corresponding homogeneous equation. Thus we already know how to find y_h (Type III).
To find y_p:

Let $y_p = De^{cx}$, substitute this into the differential equation and solve for D.

Example 2.10.

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 5e^{2x}$$

Solution:
Characteristic equation: $t^2 + 2t - 3 = 0$.
Roots: $1, -3$
Thus

$$y_h = Ae^x + Be^{-3x}$$

Let $y_p = De^{2x}$

$$y_p' = 2De^{2x}, \quad y_p'' = 4De^{2x}$$

Thus

$$\begin{aligned} 5e^{2x} &= y_p'' + 2y_p' - 3y_p \\ &= 4De^{2x} + 4De^{2x} - 3De^{2x} \\ &= 5De^{2x} \\ D &= 1 \\ y_p &= e^{2x} \end{aligned}$$

Hence the general solution is :

$$y = \underbrace{Ae^x + Be^{-3x}}_{y_h} + \underbrace{e^{2x}}_{y_p}$$

Example 2.11. Solve the initial value problem:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^x$$

$$y(0) = 0,\ y'(0) = 1$$

Solution: Characteristic equation: $t^2 - 5t + 6 = 0$.
Roots: $2, 3$
$y_h = Ae^{2x} + Be^{3x}$.
$y_p = De^x$
$y_p' = y_p'' = De^x$
Thus

$$\begin{aligned} e^x &= y_p'' - 5y_p' + 6y_p \\ &= (D - 5D + 6D)e^x \\ &\implies D = 1/2 \end{aligned}$$

So the general solution is:

$$y = Ae^{2x} + Be^{3x} + \frac{1}{2}e^x$$

Now use the initial conditions to solve for A and B:

$$\begin{aligned} 0 = y(0) &= A + B + 1/2 \\ y'(x) &= 2Ae^{2x} + 3Be^{3x} + \frac{1}{2}e^x \\ 1 = y'(0) &= 2A + 3B + 1/2 \end{aligned}$$

$$A = -2, B = 3/2$$

Answer:

$$y = -2e^{2x} + \frac{3}{2}e^{3x} + \frac{1}{2}e^x$$