CHAPTER 3: EXPONENTS AND POWER FUNCTIONS

1. The algebra of exponents

1.1. **Natural Number Powers.** It is easy to say what is meant by a^n – 'a (raised to) to the (power) n – if $n \in \mathbb{N}$. For example:

$$2^3 = 2 \cdot 2 \cdot 2 = 8$$
$$3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$$

In general, if $n \in \mathbb{N}$ and $a \in \mathbb{R}$

$$a^n = \underbrace{a \cdot a \cdots a}_{n}$$

Terminology: In the expression a^b, b is said to be *the exponent* and a the *base*.
Note what happens to the exponents if we multiply two powers *with the same base*. For example:

$$a^3 \cdot a^4 = (a \cdot a \cdot a) \cdot (a \cdot a \cdot a \cdot a) = a^7 = a^{3+4}$$

In general,

$$\boxed{a^{b_1} \cdot a^{b_2} = a^{b_1+b_2}} \text{ (The First Law of Exponents)}$$

Thus, when we multiply two powers with the same base we must *add* the exponents.

1.2. **Zero and negative powers. Notation:** $a^0 = 1$ for any $a \in \mathbb{R}, \neq 0$.

Example 1.1. $10^0 = 1$, $187^0 = 1$, $(-\pi)^0 = 1$, $(1.0529)^0 = 1$.

Why is this the correct meaning to give to the symbol a^0? We would like the First Law of Exponents to continue to remain true when one or both of the exponents are equal to 0. But then, for any $a \neq 0$, we must have: $a^0 \cdot a^b = a^{0+b} = a^b = 1 \cdot a^b$ and thus (cancelling a^b on both sides) we are *forced* to conclude that $a^0 = 1$.
What about assigning a meaning to a^{-b}? Once again the definition is forced on us (see below) if we want the First Law of Exponents to continue to hold:
Notation:

$$a^{-b} = \frac{1}{a^b} \qquad (a \neq 0)$$

Example 1.2.

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

Example 1.3.

$$\left(\frac{1}{3}\right)^{-1} = \frac{1}{1/3} = 3$$

[Why is this the correct definition?
$a^{-b} \cdot a^b = a^{-b+b} = a^0 = 1 = \dfrac{1}{a^b} \cdot a^b$. Thus, cancelling a^b on both sides again, gives $a^{-b} = 1/a^b$.]
Note what happens to the exponent when we raise a number to a power and then raise this new number to another power. For example:

$$(a^2)^3 = a^2 \cdot a^2 \cdot a^2 = (a \cdot a) \cdot (a \cdot a) \cdot (a \cdot a) = a^6 = a^{2 \cdot 3}$$

In general we have:

$$\boxed{(a^b)^c = a^{b \cdot c}} \text{ (The Second Law of Exponents)}$$

Thus when we take a power of a power, we must *multiply* the exponents.

1.3. **Fractional Powers.** Recall that if $a > 0$, then $\sqrt{a}$ denotes the unique positive number whose square is a. More generally, $\sqrt[m]{a}$ denotes the unique positive number whose mth power is a.

Example 1.4. $\sqrt[3]{27} = 3$. $\sqrt[5]{32} = 2$. $\sqrt[4]{81} = 3$, and so on.

Note that $\sqrt[m]{a}$ is usually only defined if $a \geq 0$. However, when m is an *odd natural number*, then $\sqrt[m]{a} = a^{1/m}$ makes sense for any real number a, negative, positive or zero. For odd m, any negative number a has a unique *negative* mth root. Indeed, when m is *odd*, and $a > 0$ we have:

$$\sqrt[m]{-a} = -\sqrt[m]{a}$$

Example 1.5. $\sqrt[3]{-8} = -\sqrt[3]{8} = -2$. $\sqrt[5]{-32} = -2$, and so on.

Notation: $a^{1/m} = \sqrt[m]{a}$ if $a \geq 0$.
[Why is this the correct meaning for $a^{1/m}$? This time we find that this definition is forced on us by the requirement that the Second Law of Exponents continue to hold even when one of the exponents is a fraction:
For $a > 0$, we have

$$\left(a^{\frac{1}{m}}\right)^m = a^{\frac{1}{m} \cdot m} = a^1 = a$$

and thus $a^{1/m}$ must be the mth root of a.]

Example 1.6.

$$8^{\frac{1}{3}} = \sqrt[3]{8} = 2, \quad 81^{\frac{1}{4}} = \sqrt[4]{81} = 3$$

Example 1.7.

$$25^{-\frac{1}{2}} = \frac{1}{25^{1/2}} = \frac{1}{5}$$

Example 1.8. For any number $a > 0$,

$$\frac{1}{\sqrt[m]{a}} = \frac{1}{a^{1/m}} = a^{-\frac{1}{m}}$$

1.4. **Rational Exponents.** We can now make sense of a^q where $a > 0$ and $q \in \mathbb{Q}$.
For if $q \in \mathbb{Q}$, then $q = n/m$ for some $n \in \mathbb{Z}$, $m \in \mathbb{N}$. If the Second Law of Exponents is to continue to hold, we must have

$$a^q = a^{n/m} = \left(a^{1/m}\right)^n = \left(\sqrt[m]{a}\right)^n$$

Notation: If $a > 0$ and if $q \in \mathbb{Q}$, $q = n/m$ then $a^q = (\sqrt[m]{a})^n$.
Note: This is the same as $(a^n)^{1/m} = \sqrt[m]{a^n}$.

Example 1.9.

$$8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = 2^2 = 4$$

Example 1.10.

$$4^{-\frac{5}{2}} = \left(4^{\frac{1}{2}}\right)^{-5} = 2^{-5} = \frac{1}{32}$$

Example 1.11.

$$81^{\frac{5}{4}} = \left(81^{\frac{1}{4}}\right)^5 = 3^5 = 243$$

1.5. **Real number exponents.** We have now given a meaning to the notation a^q for any positive number a and any *rational* number q, and in such a way that the two laws of exponents holed for rational powers.
How can we give a meaning to the expression a^r if r is an *irrational* real number? What should be the meaning of $5^{\sqrt{2}}$ or $\pi^{\sqrt[3]{7}}$, for instance?
This is a much more difficult problem To deal with properly would involve a deeper understanding of the real line at a level more appropriate to an Honours course.
I will simply state, without proof, the following fact:
It is possible to give a definite meaning to a^r for any $a > 0$ and any *real number* r – rational or irrational – in one and only one way so that

(1) It has the meaning assigned above if $r \in \mathbb{Q}$.
(2) The two 'laws of exponents' continue to hold for all real exponents.
(3) For fixed $a > 0$, the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = a^x$ is *continuous.*

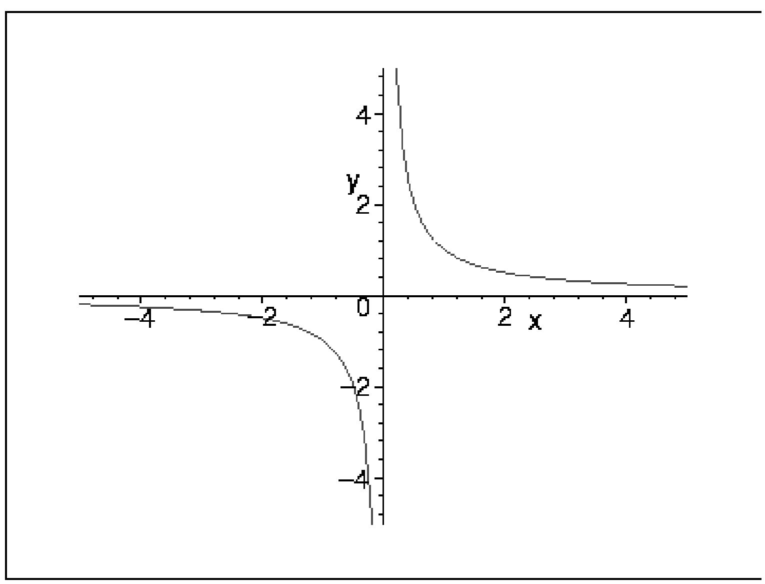
We will deal with the notion of a *continuous function* later. For the present, it is enough to say that this last condition means that the graph of the function $f(x) = a^x$ is an unbroken curve and that if r is a real number and q is a good rational approximation to r, then a^q is a good approximation to a^r.
Note that any scientific calculator has a $\boxed{y^x}$ button, which will calculate a^r where a is positive and r is any real number which the calculator can accept as input. Of course, most of the time the calculator will only return a finite decimal (and hence rational) *approximation* of the true value.

2. Power Functions

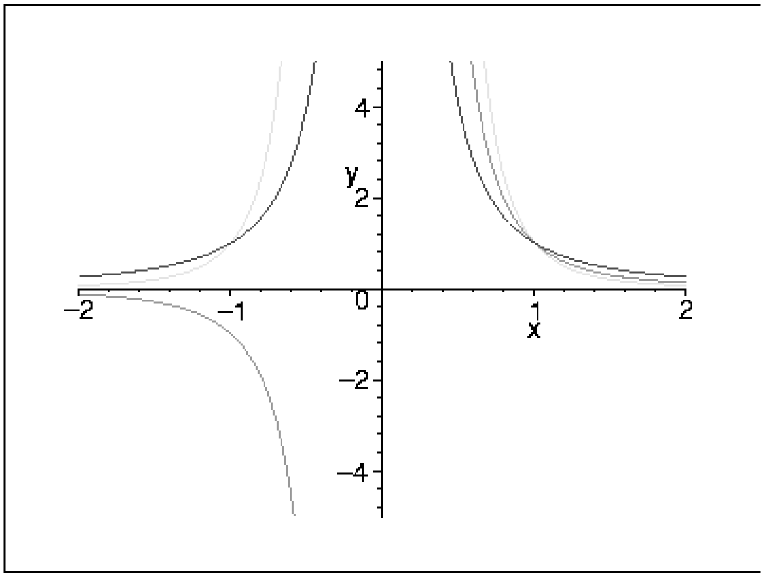
Definition 2.1. A *power function* is a function of the form $f(x) = x^r$ for some fixed real number r. Note that, for positive r, the domain of this function is $[0, \infty)$ unless r is a rational number with *odd* denominator. (When r is negative, 0 does not lie in the domain – why?)

Example 2.1. The functions $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$, etc. – whose graphs we have already considered – are power functions.

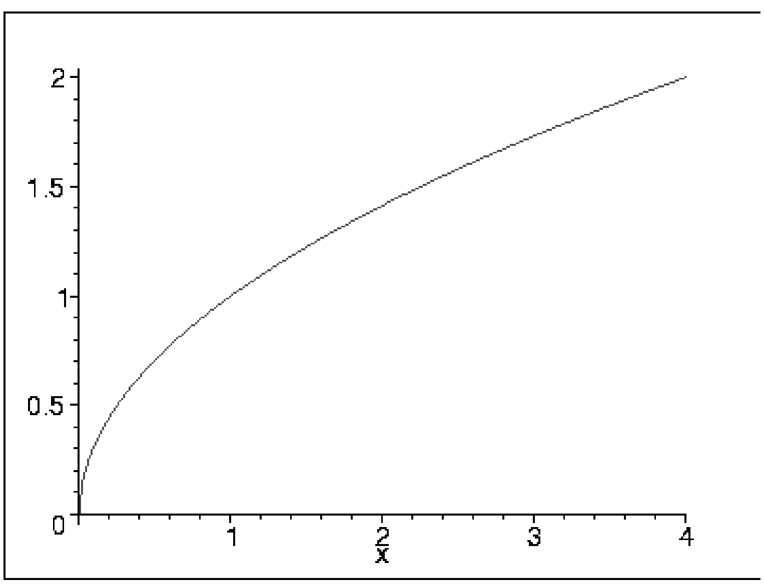
Example 2.2. The function $f(x) = 1/x$ is a power function; it is just the function $f(x) = x^{-1}$ (so $r = -1$ here). The domain is $(-\infty, 0) \cup (0, \infty)$. Its graph is a *hyperbola*:



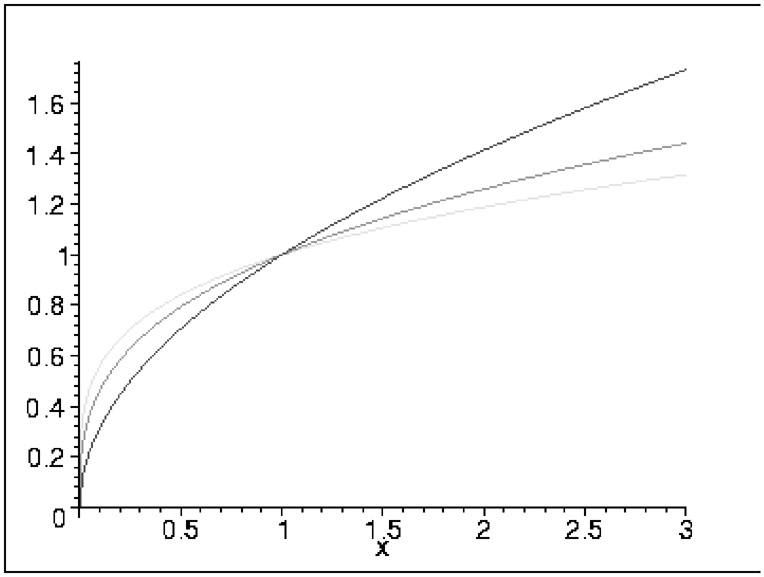
Example 2.3. More generally, the function $f(x) = 1/x^m$ ($m \in \mathbb{N}$) is just the power function $f(x) = x^{-m}$ (with $r = -m$). Its domain is $(-\infty, 0) \cup (0, \infty)$.
For example, here are the graphs of $f(x) = 1/x^2$ and $f(x) = 1/x^3$ (which is which?):



Example 2.4. The function $f(x) = \sqrt{x}$ is just the power function $f(x) = x^{1/2}$. The domain is $[0, \infty)$.



Example 2.5. More generally, the function $f(x) = \sqrt[m]{x}$ is just the power function $f(x) = x^{1/m}$. Its domain is $[0, \infty)$ if m is even and $\mathbb{R}$ if m is odd. Here are the graphs of $f(x) = x^{1/2}$ and $f(x) = x^{1/3}$ (which is which?):



Example 2.6. The function $f(x) = 1/\sqrt[m]{x}$ is just the power function $f(x) = x^{-1/m}$.