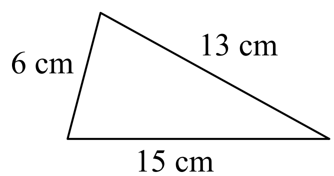
8.2 The Law of Cosines

The Law of Cosines is used when you do know either an angle and two adjacent sides, or all three sides and no angle. The following is the formula for Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$
$$b^2 = a^2 + c^2 - 2ac \cos \beta$$
$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

Law of Cosines can be used to find an angle given the lengths of all three sides, or can be used to find the side opposite a given angle and its adjacent sides.

Example Solve the triangle. Round measurements to the nearest tenth.



Applying the Law of Cosines, solve for the largest angle first. The largest angle is opposite the longest side.

$a^2 = b^2 + c^2 - 2bc \cos \alpha$
$15^2 = 6^2 + 13^2 - 2(6)(13) \cos \alpha$
$225 = 36 + 169 - 156 \cos \alpha$
$225 = 205 - 156 \cos \alpha$
$20 = 156 \cos \alpha$
$-\frac{20}{156} = \cos \alpha$
$\alpha = \cos^{-1}\left(-\frac{20}{156}\right) = 97.4^\circ$

α =97.4°

Now that we know the measure of one angle, we can apply the Law of Sines to solve for one angle. The best angle to solve for is the next largest angle. Since 13 is the next longest side, then the angle opposite represents the next largest angle.

$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$
$\frac{15}{\sin 97.4^\circ} = \frac{13}{\sin \beta} = \frac{6}{\sin \gamma}$
$\frac{15}{\sin 97.4^\circ} = \frac{13}{\sin \beta}$
$\sin \beta = \frac{13 \sin 97.4^\circ}{15} = 0.8594$
$\beta = \sin^{-1} 0.8594 = 59.3^\circ$

When calculating the inverse sine, use the entire value not just the 4 digits, as shown here.

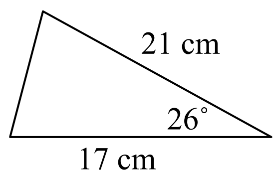
β =59.3°

Now that we know two angles, we can use the idea that the sum of the angles of a triangle equals 180 degrees.

$\alpha + \beta + \gamma = 180^\circ$
$97.4^\circ + 59.3^\circ + \gamma = 180^\circ$
$\gamma = 180^\circ - (97.4^\circ + 59.3^\circ)$
$\gamma = 180^\circ - (156.7^\circ) = 23.3^\circ$

$\boxed{\gamma = 23.3^\circ}$

Example Solve the triangle. Round measurements to the nearest tenth.



Notice this time we do not know the lengths of all three sides of the triangle. But, we do know the measure of an angle and the lengths of its' adjacent sides. We can apply the Law of Cosines to find the length of the missing side.

$a^2 = b^2 + c^2 - 2bc \cos \alpha$
$a^2 = 21^2 + 17^2 - 2(21)(17) \cos 26^\circ$
$a^2 = 441 + 289 - 714 \cos 26^\circ$
$a^2 = 730 - 641.74$
$a^2 = 88.26$
$a = 9.4$

$\boxed{a = 9.4}$

When calculating the length of side *a*, use the entire value when taking the square root not just the 4 digits, as shown here.

Now that we know the lengths of all three sides and an angle, we can apply the Law of Sines to continue solving the triangle.

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$
$$\frac{9.4}{\sin 26^\circ} = \frac{17}{\sin\beta} = \frac{21}{\sin\gamma}$$

Since 21 is the largest side, let's begin by solving for γ.

$$\frac{9.4}{\sin 26^\circ} = \frac{21}{\sin\gamma}$$
$$\sin\gamma = \frac{21\sin 26^\circ}{9.4} = 0.9793$$
$$\gamma = \sin^{-1} 0.9793 = 78.3^\circ$$

$\boxed{\gamma = 78.3^\circ}$

Now that we know two angles, we can use the idea that the sum of the angles of a triangle equals 180 degrees.

$\alpha + \beta + \gamma = 180^\circ$
$26^\circ + \beta + 78.3^\circ = 180^\circ$
$\beta = 180^\circ - (26^\circ + 78.3^\circ)$
$\beta = 180^\circ - 104.3 = 75.7^\circ$

$\beta = 75.7^\circ$

Try the following:

1. Solve ΔABC given a = 86 inches, b = 32 inches, and c = 53 inches. Round measurements to the nearest unit.

2. Solve ΔABC given $\beta = 57.3^\circ$, a = 6.08 cm, and c = 5.25 cm. Round angle measurements to the nearest tenth and lengths to the nearest hundredths.

3. Solve ΔABC given a = 23.4 m, b = 6.9 m, and c = 31.3 m. Round angle measurements to the nearest tenths.

4. Solve ΔABC given $\gamma = 58.4^\circ$, b = 7.23 m, and c = 6.54 m. Round angle measurements to the nearest tenth and lengths to the nearest hundredths.

Answers: no solution ; $b = 5.48$, $\alpha = 69.0^\circ$, $\gamma = 53.7^\circ$; no solution ;
triangle 1- $\beta = 70.3^\circ$, $\alpha = 51.3^\circ$, $a = 5.99$; triangle 2 - $\beta = 109.7^\circ$, $\alpha = 11.9^\circ$, $a = 1.58$