Part 10: Further AC Theory

10.1 Resistive And Inductive Series Circuit (RL Circuit)

In part 9 we looked at circuits that contain only resistance, inductance or capacitance. However we also mentioned that circuits can not, in practice, be solely inductive. Therefore we must take account of circuits that have resistance and inductance and we can represent a coil as a pure inductor in series with a resistor (figure 10.1a). In phasor diagrams it is normal to take the resistive element of the circuit as the horizontal and in this case, because it is a series circuit, the current must be common to both resistor and inductor, therefore it is taken as a reference for the phasor diagram (shown in figure 10.1b).

From figure 10.1b, the voltage and current must be in phase for a resistor, so that the PD U_R across the resistor must be in phase with the current I. Current lags voltage by 90° in a pure inductor, so that U_L (the PD across the inductor) is drawn vertically up from the current phasor (i.e. tU_L leads the current by 90°). The supply voltage (U) is the combination of both these potential differences and is found from the phasor sum of U_R and U_L, using the parallelogram in figure 4.2b. The angle between the supply voltage and the current is called the *phase angle* (ϕ). In figure 10.1c we can see that for circuits with resistance and inductance in series the current lags the supply voltage; the greater the inductance (and therefore the greater U_L) the greater the lag. Thus in this case, ϕ is called a lagg*ing phase angle* (quoted as ϕ° lagging).

Note that figure 10.1c shows an alternative construction of the phasor diagram. Also, in figures 10.1b and c, U_L is shown above U_R because the voltage across the inductor leads the current and therefore leads the voltage across the resistor as well. The phasor is assumed to rotate anticlockwise at ω rads/sec.

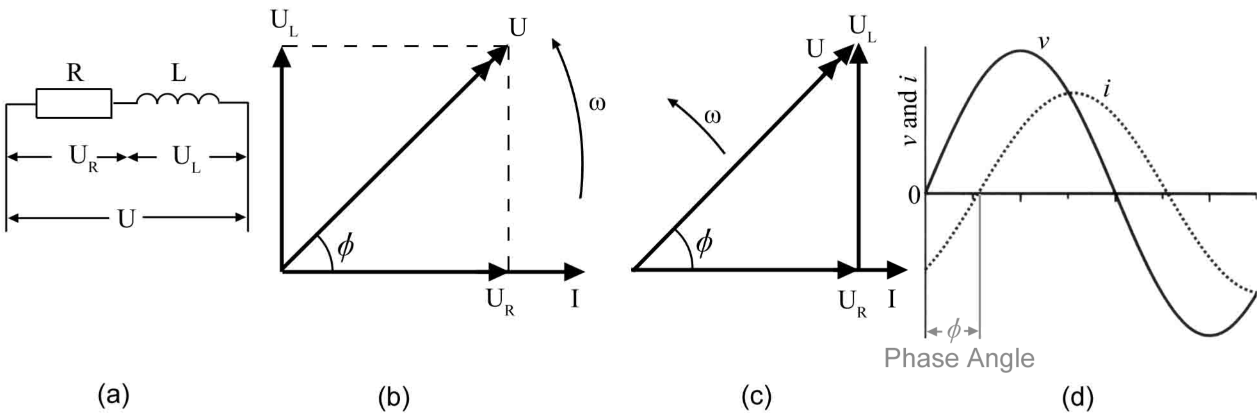

Figure 10.1: (a) The resistive and inductive circuit; (b) the phasor parallelogram; (c) the phasor triangle; (d) the wave diagram. Note that *i* and *v* are not plotted on the same scale.

It can be seen from figure 10.1c that the voltages U, U_R and U_L together form a right angled triangle, to which Pythagoras' theorem applies. Thus:

$$U^2 = U_R^2 + U_L^2$$

Simple trigonometry also applies, so that: $$\cos\phi = \frac{U_R}{U} \quad \sin\phi = \frac{U_L}{U} \quad \tan\phi = \frac{U_L}{U_R}$$

For a resistor, the effect limiting the current: $$U_R = IR$$

For a pure inductor, the effect limiting current is (section 9.5): $U_L = IX_L$

In a circuit that has both resistance and inductance, both effects contribute to limiting the voltage, and:

$$U^2 = U_R^2 + U_L^2 = (IR)^2 + (IX_L)^2 = I^2(R^2 + X_L^2)$$

$$\frac{U^2}{I^2} = R^2 + X_L^2 \quad \text{and} \quad \frac{U}{I} = \sqrt{(R^2 + X_L^2)}$$

For an AC circuit, the ratio U/I is know as *impedance* (Z) which has the units of ohms (Ω). The impedance in ohms may be defined as the applied voltage in volts needed to drive a current of one ampere. Therefore:

$$\mathbf{Z} = \frac{\mathbf{U}}{\mathbf{I}} = \sqrt{(\mathbf{R}^2 + \mathbf{X}_L^2)} \quad \text{and} \quad \mathbf{Z}^2 = \mathbf{R}^2 + \mathbf{X}_L^2$$

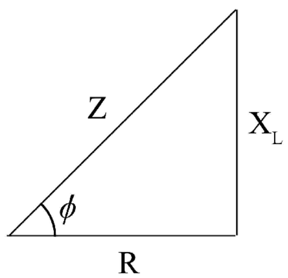

Figure 10.2: The impedance for resistive and inductive AC series circuits.

We can see that this latter formula can apply to a right-angled triangle. Such an triangle is called an impedance triangle and is shown in figure 10.2. The angle ϕ inclined between sides R and Z is the same as the phase angle between current and voltage for a given circuit. Applying simple trigonometry:

$$\cos\phi = \frac{\mathbf{R}}{\mathbf{Z}} \quad \sin\phi = \frac{\mathbf{X}_L}{\mathbf{Z}} \quad \tan\phi = \frac{\mathbf{X}_L}{\mathbf{R}}$$

Example

A series circuit consisting of an inductor of negligible resistance and a pure resistor of 12Ω is connected to a 30V, 50Hz AC supply. If the current is 2A, calculate:

(a) the PD across the resistor
(b) the PD across the inductor
(c) the inductance of the inductor
(d) the phase angle between applied voltage and current.
(e) the impedance of the circuit

(a) since the resistor obeys Ohm's law, $U_R = IR = 2 \times 12 = 24V$

(b) $U^2 = U_R^2 + U_L^2$, therefore, $U_L^2 = U^2 - U_R^2 = 30^2 - 24^2 = 324V^2$

$U = \sqrt{324} = 18V$

(c) $X_L = \frac{U_L}{I} = \frac{18}{12} = 9\Omega$

(d) $X_L = 2\pi f L$, so $L = \frac{X_L}{2\pi f} = \frac{9}{2\pi \times 50} = 0.0286\text{H}$ or 28.6mH

$\tan\phi = \frac{U_L}{U_R} = \frac{18}{24} = 0.75$ therefore, $\phi = 36.9°$

(e) $Z = \sqrt{(R^2 + X_L^2)} = \sqrt{(12^2 + 9^2)} = \sqrt{255} = 15\Omega$

$I = \frac{U}{Z} = \frac{30}{15} = 2\text{A}$

10.2 Resistive And Capacitive Series Circuit (RC Circuit)

Some types of capacitor have considerable current leakage and the affect of this is similar to that of a resistor in series with the capacitor. Even if the leakage current is small, a resistor is often connected in series with a capacitor and the circuit arrangement is shown in figure 10.3b. As with figure 10.2 the current is taken as the reference (horizontal) phasor (figure 10.3b and c). The PD U_R across the resistor will be in phase with this current, while the current will lead the PD U_C across the capacitor by 90° (i.e the U_C leads the current). The supply voltage is the phasor sum of the component voltages U_R and U_C, and is found by completing the parallelogram as in figure 10.3b or by completing the phasor triangle as in figure 10.3c.

As in figure 10.2 the angle between the supply voltage and the current is the *phase angle* (ϕ). Since in this case current leads the supply voltage, the angle is called a *leading phase angle* (quoted as ϕ° leading). Figure 10.3d shows the wave diagram for the circuit.

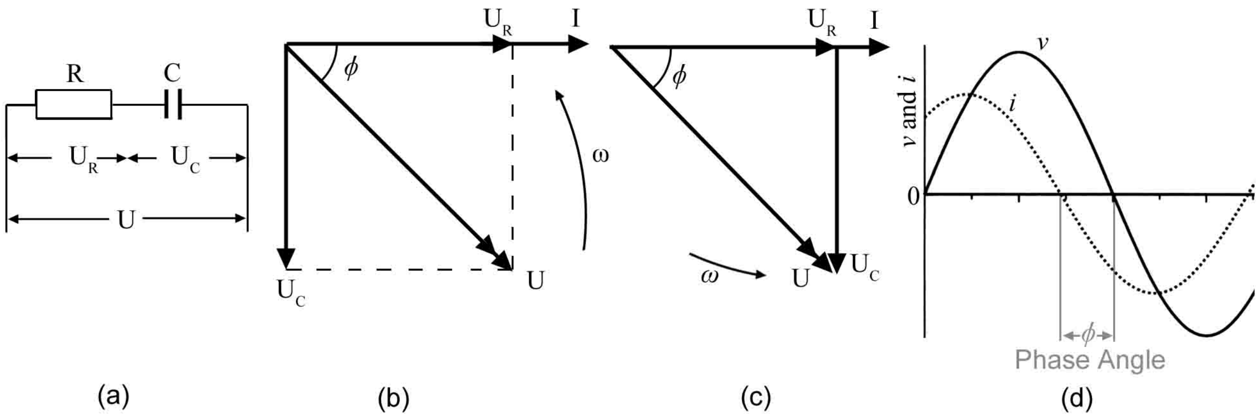

Figure 10.3: (a) The resistive and capacitive circuit; (b) the phasor parallelogram; (c) the phasor triangle; (d) the wave diagram. Note that *i* and *v* are not plotted on the same scale.

From figure 10.3c it is seen that the voltages U, U_R and U_L together form a right angled triangle, to which Pythagoras' theorem applies. Thus:

$$U^2 = U_R^2 + U_C^2$$

Simple trigonometry also applies, so that: $\mathbf{\cos\phi = \frac{U_R}{U}}$ $\mathbf{\sin\phi = \frac{U_C}{U}}$ $\mathbf{\tan\phi = \frac{U_C}{U_R}}$

The impedance of the resistive-capacitive series circuit can be found from:

$$U^2 = U_R^2 + U_C^2$$

$$(IZ)^2 = (IR)^2 + (IX_C)^2$$

Therefore (figure 10.4): $\mathbf{Z} = \dfrac{\mathbf{U}}{\mathbf{I}} = \sqrt{\left(\mathbf{R}^2 + \mathbf{X_C}^2\right)}$ and $\mathbf{Z}^2 = \mathbf{R}^2 + \mathbf{X_C}^2$

Once again, the impedance circuit can be defined as the applied voltage needed to drive a current of one ampere through it. Also, once again we can draw an impedance triangle (figure 10.4) and:

$$\cos\phi = \frac{\mathbf{R}}{\mathbf{Z}} \quad \sin\phi = \frac{\mathbf{X_C}}{\mathbf{Z}} \quad \tan\phi = \frac{\mathbf{X_C}}{\mathbf{R}}$$

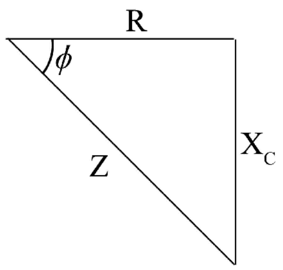

Figure 10.4: The impedance for resistive and capacitive AC series circuits.

10.3 General Series Circuit (RLC Circuits)

A circuit having resistance, inductance and capacitance in series is known as a general series circuit. Such a circuit is shown in figure 10.5a, figure 10.5b shows the phasor diagram and figure 10.5c shows the impedance diagram. The current is taken as a reference since the same current passes though all three components. The PD U_R across the resistor is in phase with the current, while the the current lags U_L (the PD across the inductor) and the leads U_C (the PD across the capacitor), both by 90°. U_L and U_C are thus 180° out of phase and directly oppose each other, so the effective PD that is out of phase with U_R is the difference between them. Thus:

$$U^2 = U_R^2 + (U_L - U_C)^2$$

$$(IZ)^2 = (IR)^2 + (IX_L - IX_C)^2$$

$$\mathbf{Z = \sqrt{[R^2 + (X_L - X_C)^2]}}$$

From figure 10.5b:

$$\cos\phi = \frac{\mathbf{U_R}}{\mathbf{U}} \quad \sin\phi = \frac{\mathbf{U_L - U_C}}{\mathbf{U}} \quad \tan\phi = \frac{\mathbf{U_L - U_C}}{\mathbf{U_R}}$$

The impedance diagram is shown in figure 10.5c and the inductive and capacitive impedances are drawn in different directions, their difference ($X_L - X_C$) becomes the effective circuit reactance. Thus (from figure 10.5c):

$$\cos\phi = \frac{R}{Z} \quad \sin\phi = \frac{X_L - X_C}{Z} \quad \tan\phi = \frac{X_L - X_C}{R_R}$$

The equations for sin ϕ and tan ϕ are useful because they infer a positive or negative value on ϕ. In figure 10.5 X_L is greater than X_C so that U_L exceeds U_C, therefore the current (and U_R) lags the difference between these two values. Hence, the circuit current lags the supply voltage (U) and ϕ is positive. Had X_C been greater that X_L, so that U_C exceeds U_L, the difference would lag U_R, therefore the circuit current would lead the supply voltage and ϕ would be negative. The equations for cos ϕ do not do this.

Note that because we took the current (and thus U_R) as the reference, we say that the current lags or leads the supply voltage (U) and not that the supply voltage lags or leads the current. This may seem complicated but just remember that:

- I and U_R always lag U_L, so that U_L is drawn up the vertical,
- I and U_R always lead U_C, so that U_C is drawn down the vertical,
- the sin and tan equations take U_L away from U_C so that if: $U_L - U_C > 0$, ϕ = positive and U is drawn above the the horizontal; $U_L - U_C < 0$, ϕ = negative and U is drawn below the the horizontal,
- the same applies from the sin and tan equations that use the term $X_L - X_C$.

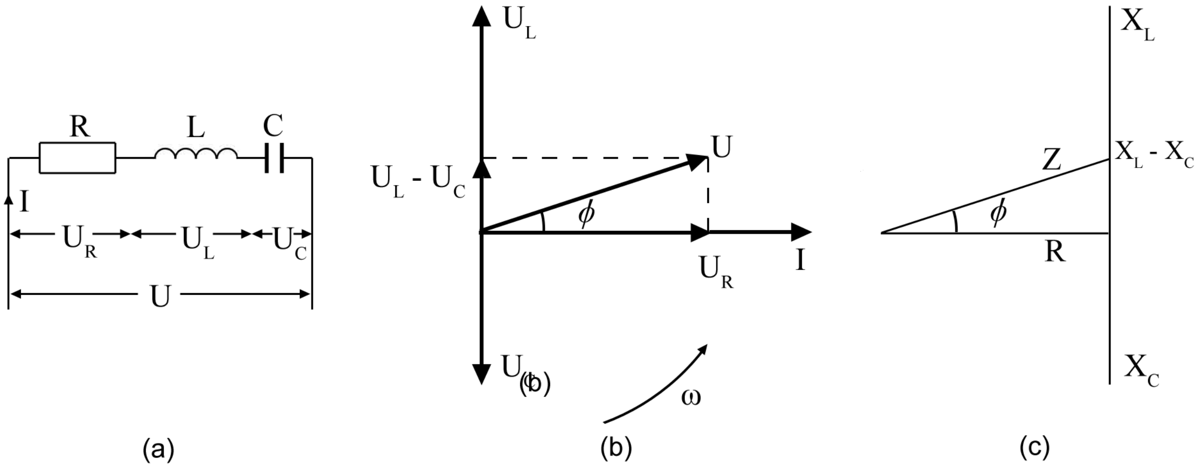

Figure 10.5: The general series circuit (a) the circuit diagram; (b) the phasor diagram; (c) the impedance triangle.

<u>Example</u>

A choke of inductance 0.318 H resistance 30Ω is connected in series with 53μF capacitance across a 24V, 50Hz supply. Calculate the circuit impedance, current and phase angle. Draw scale phasor and impedance diagrams.

$X_L = 2\pi fL = 2\pi \times 50 \times 0.318 = 100\Omega$

$$X_C = \frac{10^6}{2\pi\, f\, C'} = \frac{10^6}{2\pi \times 50 \times 53} = 60\Omega$$

$Z = \sqrt{[R^2 + (X_L - X_C)^2]} = \sqrt{[30^2 + (100 - 60)^2]} = 50\Omega$

$$I = \frac{U}{Z} = \frac{24}{50} = 0.48A$$

$U_L = IX_L = 0.48 \times 100 = 48V$

$U_C = IX_C = 0.48 \times 60 = 28.8V$

$U_R = IR = 0.48 \times 30 = 14.4V$

$$\tan v = \frac{U_L - U_C}{U_R} = \frac{48 - 28.8}{14.4} = 1.33 \qquad \phi = 53.1° \text{ lagging}$$

Since ϕ comes out as a positive value and we have taken the current as a reference, the current lags U and we say the phase angle is 53.1° lagging. The phasor and impedance diagrams are shown in figure 10.6.

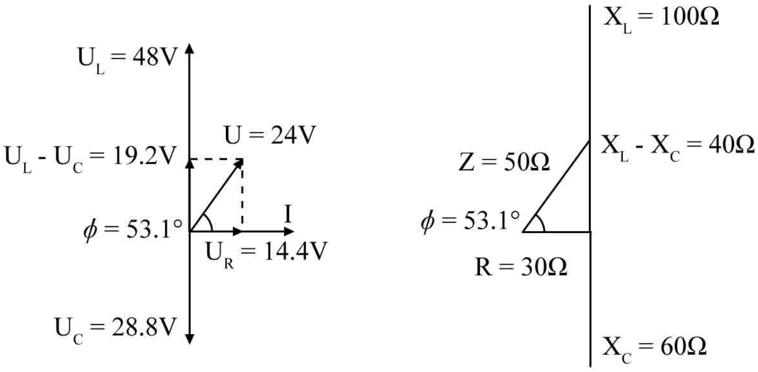

Figure 10.6:

If, in the previous example, the frequency of the supply is reduced to 25Hz the values will change and the new phasor and impedance diagrams are shown in figure10.7. Note that the change in frequency has halved the inductive reactance and doubled the capacitive reactance. Because of this the current now leads the supply voltage.

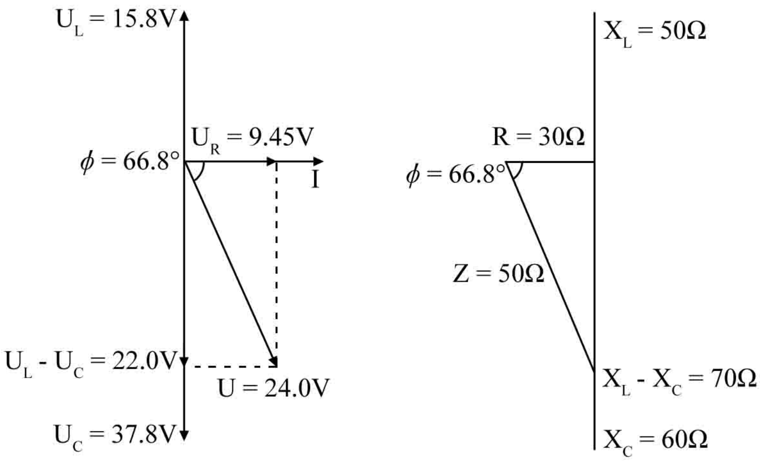

Figure 10.7:

10.4 Series Resonance

Look again at the last example. With a 50Hz supply voltage of 24V, the PD across the inductive component of the choke was 48V, while that across the capacitor was 28.8V. With a 25Hz supply, the PD across the capacitor was 37.8V. At first sight this appears impossible. However, as can be seen from figure 10.7, while both reactances are high they act against each other, and the impedance is smaller than both. Since it is the impedance that limits the current, a larger current will flow in this combined circuit, than if the circuit contained only the capacitor or the inductor in series with the resistor. This larger current passes through the resistor and the capacitor producing high potential differences across them, but as these voltages oppose each other the supply voltage can be smaller than both or one of them.

This effect leads to *series resonance*. True resonance occurs when the inductive and capacitive reactances cancel each other out exactly ($X_L = X_C$) and the current is limited only by the resistance. Mathematically, when $X_L = X_C$:

$$Z = \sqrt{[R^2 + (X_L - X_C)^2]} = \sqrt{[R^2 + 0^2]} = \sqrt{R^2} = R$$

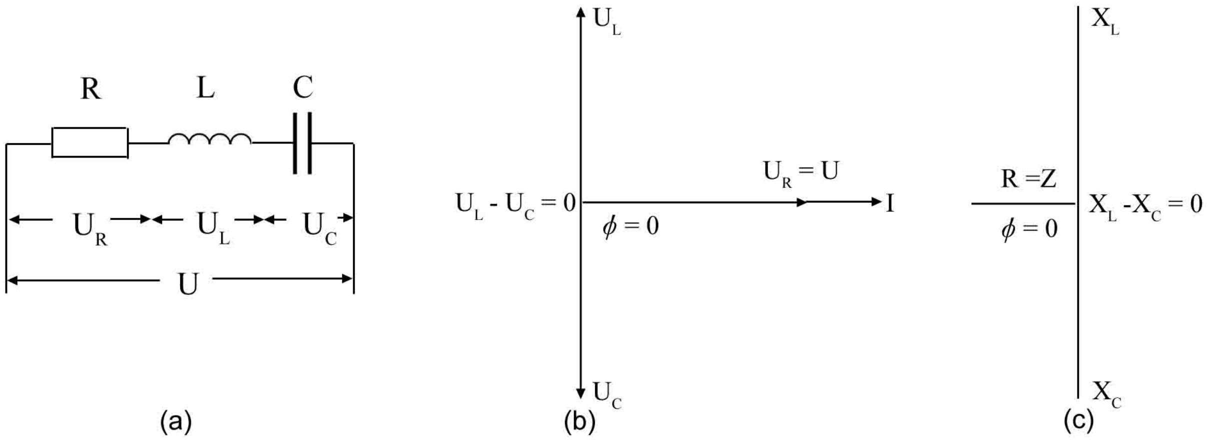

Figure 10.8: Series resonance (a) circuit diagram; (b) phasor diagram; (c) impedance triangle.

Figure 10.8 shows that at resonance a series circuit has the following properties:

i. The current and supply voltage are in phase so that $\phi = 0°$.

ii. The circuit impedance Z will be equal to the circuit resistance R, and will thus be at its maximum possible value due to the cancellation of reactances.

iii. The current will thus be at a maximum value due to minimum impedance. The series-resonant effect, with inductive and capacitive reactances equal and opposite, may be brought about in a number of ways:

   (a) Change in inductance, giving a proportional change in inductive reactance (note that $X_L = 2\pi f L$, so $X_L \alpha L$ if f is constant).

   (b) Change in capacitance, giving an inversely proportional change in capacitive reactance.

$$\text{note that } X_C = \frac{1}{2\pi f C} \quad \text{so} \quad X_C \alpha \frac{1}{C} \quad \text{if } f \text{ is constant}$$

   (c) Change in frequency. If L and C are constant, $X_L \alpha f$ and $X_C \alpha 1/f$, so an increase in frequency will increase inductive reactance and decrease capacitive reactance.

Figure 10.9a shows a number of values all plotted against frequency. R remains unaffected by frequency and is plotted as a horizontal line, X_L increases linearly with frequency and X_C reduces with frequency. At the point where the X_L and X_C curves intersect, inductive and capacitive reactance become equal but opposite and cancel, leaving the circuit resistance alone to limit the current. The frequency at which this happens is called the resonance frequency (f_r). Figure 10.9b shows how the current varies with f_r. A formula from which the resonant frequency may be determined is:

$$f_x = \frac{1}{2\pi\sqrt{(LC)}} = \frac{10^6}{2\pi\sqrt{(LC')}}$$

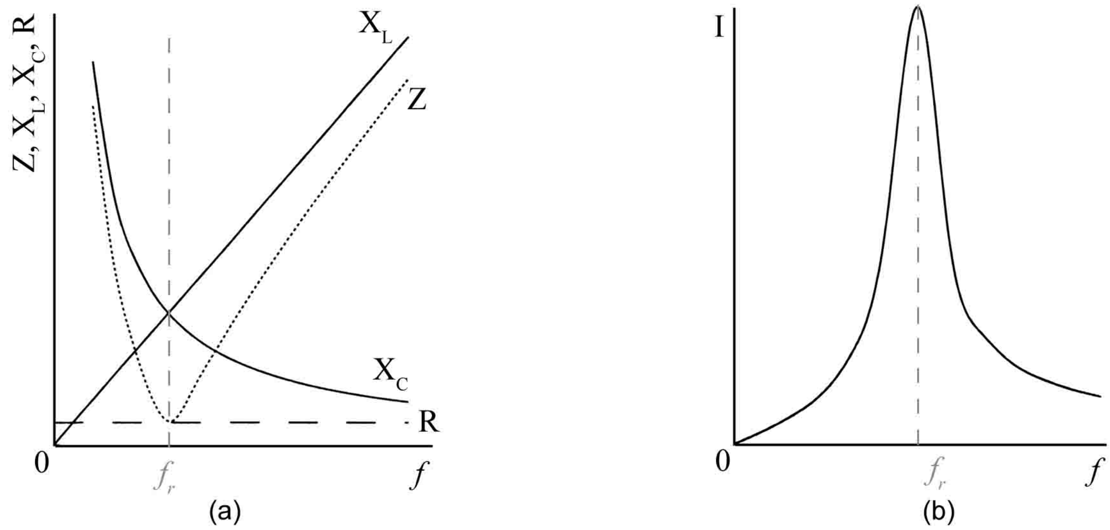

Figure 10.9: Resonance frequency (f_r) of a series circuit; (a) reactance, resistance and impedance vrs frequency; (b) circuit current vrs. frequency.

10.5 AC Parallel Circuits

There are many possible arrangements of parallel AC circuits. The easiest way to analyse these circuits is to calculate the current in each branch, then the take the supply voltage as a reference since in parallel circuits it will be the same for each branch , finally work out the phase relationship between the total current and the supply voltage.

A word of caution may be needed here, since L and C may appear to change places. In the series circuits we took the circuit current (I) as a reference, which was common to all components, and found the phasor sum of the voltages. Thus the phasor diagram contained sun of U_R, U_L and U_C; because I lags U_L, U_L is drawn vertically up and because I leads U_C, U_C is drawn vertically down. However, the supply voltage (U) is now the reference because it is common to all branches in the circuit, therefore the current of branches which have a net capacitance will lead U (being drawn above it) and the current in branches which have a net inductance will lag U (being drawn below it)

<u>Example</u>
A 60Ω resistor, a pure 0.382H inductor and a 66.3μF capacitor are connected in parallel to a 240V 50Hz supply (figure 10.10a). Calculate the current taken from the supply and its phase angle to the supply voltage. First calculate the current in each branch:

$$I_R = \frac{U}{R} = \frac{240}{60} = 4\text{A in phase with the voltage}$$

$X_L = 2\pi f L = 2 \times 3.142 \times 0.382\Omega = 120\Omega$

$$I_L = \frac{U}{X_L} = \frac{240}{120} = 2\text{A lagging voltage by } 90°$$

$$X_C = \frac{10^6}{2\pi f C'} = \frac{10^6}{2 \times 3.142 \times 50 \times 66.3} = 48\Omega$$

$$I_C = \frac{U}{X_C} = \frac{240}{48} = 5A \quad \text{leading voltage by } 90°$$

These currents drawn to scale are shown in figure 10.10b, I_C is draw vertically up because in a capacitor current leads voltage by 90°, similarly I_L is draw vertically down because in an inductor current lags voltage by 90°. Since I_L is directly opposing I_C, their difference $I_C - I_L$ is $5 - 2 = 3A$ as shown. This current is added to I_R by completing the parallelogram, and the phasor I_T representing the current from the supply is draw in. By measurement:

$I_T = 5A$, and $\phi = 36.9°$.

Thus the current is 5A, leading the supply voltage by 36.9°.

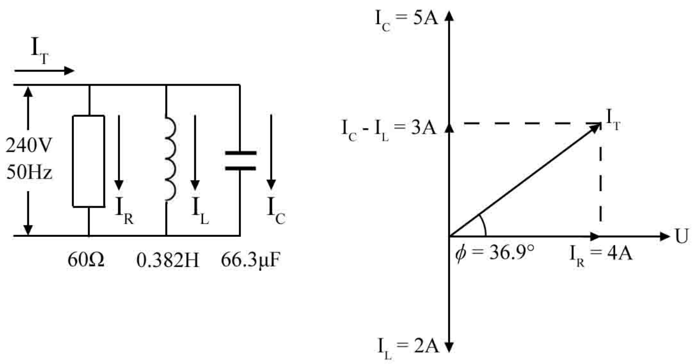

Figure 10.10:

Example

A circuit consists of a choke of resistance 15Ω and inductance 7.96nH, connected in parallel with a 7.96μF capacitor, across a 125V, 400Hz supply (Figure 10.11). Calculate the current in each branch, and the current from the supply with its phase relative to supply voltage.

In the previous example there were three different currents with either 90° or 180° between them. This example is more complex because the current in the resistance-inductance series branch will not be in phase with or at right angles to the voltage. For this branch:

$$X_L = 2\pi f L = 2 \times 3.142 \times 400 \times \frac{7.96}{1000} = 20\Omega$$

$$Z_1 = \sqrt{(R^2 + X_L^2)} = \sqrt{(15^2 + 20^2)} = 25\Omega$$

$$\cos\phi = \frac{R}{Z} = \frac{15}{25} = 0.6 \quad \phi = 53.1 \quad \text{lagging, since the circuit is inductive}$$

Although the direction of the angle is not indicated from this equation we know that in an RL circuit the current always lags the voltage.

For the capacitive branch: $X_C = \frac{10^6}{2\pi f C'} = \frac{10^6}{2 \times 3.142 \times 400 \times 7.96} = 50\Omega$

$$I_2 = \frac{U}{Z_2} = \frac{U}{X_C} = \frac{125}{50} = 2.5\Omega$$

1As this branch is purely capacitive, in it the current will lead the supply voltage by 90°. The phasor sum of I_1 and I_2 is found by completing the parallelogram (figure 10.11) and by scale measurement:

I = 3.35A and $\phi = 26°$ lagging

Thus the supply current is 3.35A lagging the supply voltage by 26°.

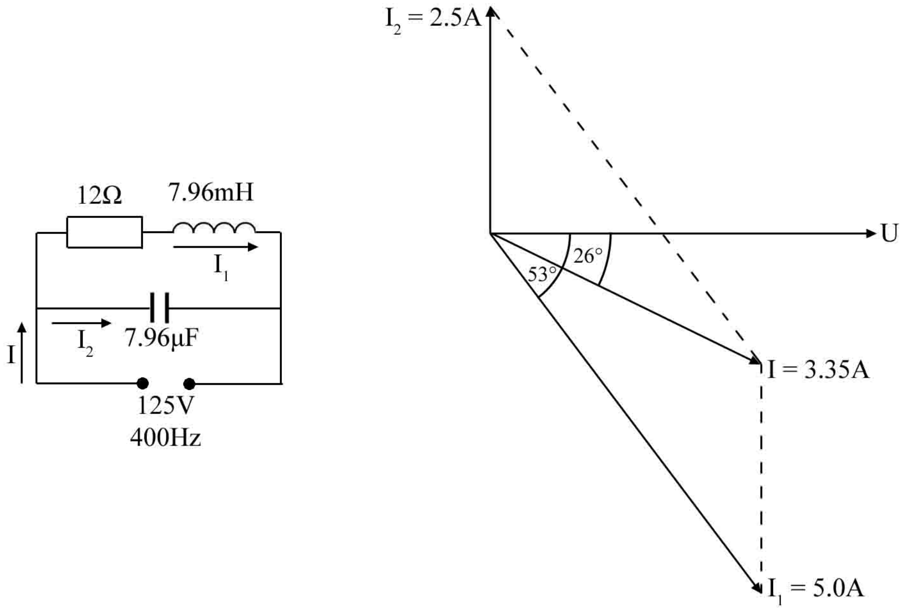

Figure 10.11:

10.6 Parallel Resonance

Consider the last example (figure 10.11) as the frequency increases:

- the capacitance will increase, the capacitive reactance therefore increases, and I_2 will increase, however the phase relative to the supply voltage will not change;
- the inductance will reduce, reducing that branch's impedance and therefore I_1 will decrease, the the angle of lag relative to the supply voltage will increase.

If I_2 increase and I_1 reduces as the frequency increases, there comes a point where the resultant current (I) comes into phase with the supply voltage. This condition is called *parallel resonance* and is illustrated in figure 10.12. Note the that the supply current is very small in comparison to I_2 and I_1 because it is their phasor sum and they operate in opposite directions. Under these conditions the only current taken from the supply is to replace losses incurred by the resistor in the inductor branch. In series resonance the current reaches a maximum at the resonance frequency, whereas in parallel circuits current reaches a minimum at the resonant frequency as the circuits impedance is a maximum at this point.

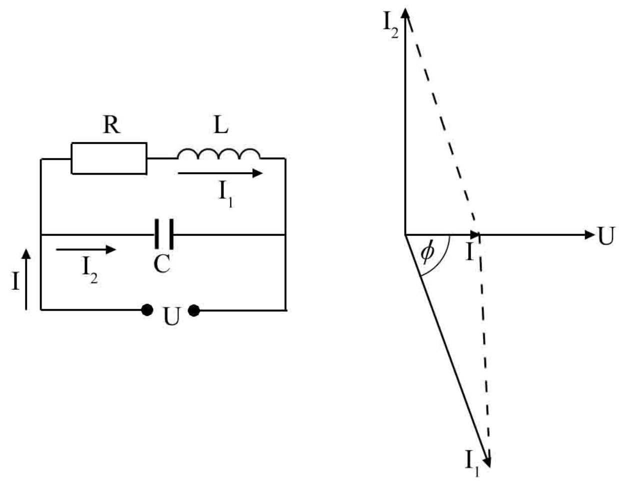

Figure 10.12: The circuit diagram and phasor diagram for resonance in a parallel AC circuit.