Difference Equations

to

Differential Equations

Section 6.6

Trigonometric Substitutions

In the last section we saw that

$$\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1}(x) + c.$$

However, we arrived at this result as a consequence of our differentiation of the arc sine function, not as the outcome of the application of some systematic approach to the evaluation of integrals of this type. In this section we will explore how substitutions based on the arc sine, arc tangent, and arc secant functions provide a systematic method for evaluating integrals similar to this one.

Sine substitutions

To begin, consider evaluating

$$\int \frac{1}{\sqrt{1-x^2}}\, dx$$

by using the substitution $u = \sin^{-1}(x)$. The motivation for such a substitution stems from the fact that, for $-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$, $u = \sin^{-1}(x)$ if and only if $x = \sin(u)$. In the latter form, we see that

$$dx = \cos(u)du$$

and

$$\sqrt{1-x^2} = \sqrt{1-\sin^2(u)} = \sqrt{\cos^2(u)} = |\cos(u)|. \qquad (6.6.1)$$

Since $\cos(u) \geq 0$ when $\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$, (6.6.1) becomes

$$\sqrt{1-x^2} = \cos(u).$$

Thus

$$\int \frac{1}{\sqrt{1-x^2}}\, dx = \int \frac{\cos(u)}{\cos(u)}\, du = \int du = u + c = \sin^{-1}(x) + c.$$

Of course, there is nothing new in the result itself; it is the technique, which we may generalize to other integrals of a similar type, which is of interest. Specifically, for an integral with a factor of the form

$$\sqrt{a^2 - x^2}$$

or

$$\frac{1}{\sqrt{a^2-x^2}},$$

where $a > 0$, the substitution

$$x = a\sin(u), \frac{\pi}{2} \leq u \leq \frac{\pi}{2}, \tag{6.6.2}$$

may prove to be useful because of the simplification

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2(x)} = \sqrt{a^2(1 - \sin^2(u))} = a\sqrt{\cos^2(u)} = a\cos(u). \tag{6.6.3}$$

Although this substitution is equivalent to the substitution $u = \sin^{-1}\left(\frac{x}{a}\right)$, we will see that it is more convenient to work with it in the form $x = a\sin(u)$.

Example To evaluate the integral $\int \frac{1}{\sqrt{9 - x^2}}\,dx$, we make the substitution

$$\begin{aligned} x &= 3\sin(u), -\frac{\pi}{2} < u < \frac{\pi}{2}, \\ dx &= 3\cos(u)du. \end{aligned}$$

Note that we omit both $u = -\frac{\pi}{2}$ and $u = \frac{\pi}{2}$ since the function being integrated is not defined at either $x = -3$ or $x = 3$. Then

$$\begin{aligned} \int \frac{1}{\sqrt{9 - x^2}}\,dx &= \int \frac{3\cos(u)}{\sqrt{9 - 9\sin^2(u)}}\,du \\ &= \int \frac{3\cos(u)}{3\sqrt{1 - sin^2(u)}}\,du \\ &= \int \frac{\cos(u)}{\sqrt{\cos^2(u)}}\,du \\ &= \int \frac{\cos(u)}{\cos(u)}\,du \\ &= \int du \\ &= u + c. \end{aligned}$$

Now $x = 3\sin(u)$ implies that $u = \sin^{-1}\left(\frac{x}{3}\right)$, so we have

$$\int \frac{1}{\sqrt{9 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{3}\right) + c.$$

Example To evaluate $\int \sqrt{4 - x^2}\,dx$, we make the substitution

$$\begin{aligned} x &= 2\sin(u), -\frac{\pi}{2} \leq u \leq \frac{\pi}{2}, \\ dx &= 2\cos(u)du. \end{aligned}$$

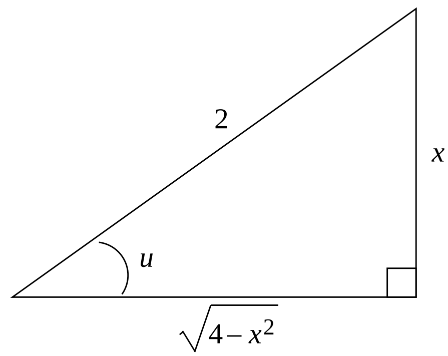

Figure 6.6.1 Right triangle with $\sin(u) = \frac{x}{2}$

Then

$$\begin{aligned}
\int \sqrt{4-x^2}\,dx &= 2\int \sqrt{4-4\sin^2(u)}\cos(u)du \\
&= 4\int \sqrt{1-\sin^2(u)}\cos(u)du \\
&= 4\int \sqrt{\cos^2(u)}\cos(u)du \\
&= 4\int \cos^2(u)du \\
&= 4\int \frac{1+\cos(2u)}{2}\,du \\
&= 2\int (1+\cos(2u))du \\
&= 2u+\sin(2u)+c \\
&= 2u+2\sin(u)\cos(u)+c.
\end{aligned}$$

Since $x = 2\sin(u)$, $\sin(u) = \frac{x}{2}$ and $u = \sin^{-1}\left(\frac{x}{2}\right)$. Moreover,

$$\cos^2(u) = 1-\sin^2(u) = 1-\frac{x^2}{4} = \frac{4-x^2}{4},$$

so

$$\cos(u) = \frac{1}{2}\sqrt{4-x^2},$$

where we have, once again, used the fact that $\cos(u) \geq 0$ since $-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$. Note that this expression for $\cos(u)$ may also be deduced from Figure 6.6.1, where we have a right triangle with an acute angle of size u such that $\sin(u) = \frac{x}{2}$. Putting everything together, we have

$$\int \sqrt{4-x^2}\,dx = 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}x\sqrt{4-x^2} + c.$$

Notice that a considerable amount of the work in the previous example involved expressing the answer in terms of x once it had been found in terms of u. The next example illustrates how this work is unnecessary when evaluating definite integrals since we can change the limits of integration and, from that point on, do all our work in terms of u.

Example Recall that, for $r > 0$, the graph of $y = \sqrt{r^2 - x^2}$ is the upper half of a circle of radius r centered at the origin. Hence we should have

$$2\int_{-r}^{r} \sqrt{r^2 - x^2}\, dr = \pi r^2.$$

We are now able to verify this. Let

$$\begin{aligned} x &= r\sin(u), -\frac{\pi}{2} \le u \le \frac{\pi}{2}, \\ dx &= r\cos(u)du. \end{aligned}$$

Then $u = \sin^{-1}\left(\frac{x}{r}\right)$, so when $x = -r$,

$$u = \sin^{-1}(-1) = -\frac{\pi}{2},$$

and when $x = r$,

$$u = \sin^{-1}(1) = \frac{\pi}{2}.$$

Thus

$$\begin{aligned} 2\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx &= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2 - r^2\sin^2(u)}\; r\cos(u)du \\ &= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2\sqrt{\cos^2(u)}\cos(u)du \\ &= 2r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(u)du \\ &= 2r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos(2u)}{2}\, du \\ &= r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos(2u))du \\ &= r^2\left(u + \frac{\sin(2u)}{2}\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= r^2\left(\left(\frac{\pi}{2}+0\right) - \left(-\frac{\pi}{2}+0\right)\right) \\ &= \pi r^2. \end{aligned}$$

Tangent substitutions

In a similar fashion, the substitution

$$x = a\tan(u), -\frac{\pi}{2} < u < \frac{\pi}{2}, \tag{6.6.4}$$

may be useful for integrals which have a factor of the form

$$\sqrt{a^2+x^2},$$

$$\frac{1}{\sqrt{a^2+x^2}},$$

or

$$\frac{1}{a^2+x^2}$$

because of the simplification

$$a^2+x^2 = a^2 + a^2\tan^2(u) = a^2(1+\tan^2(u)) = a^2\sec^2(u). \tag{6.6.5}$$

Note that with our restriction on u, this substitution is equivalent to the substitution $u = \tan^{-1}\left(\frac{x}{a}\right)$.

Example To evaluate the integral $\displaystyle\int \frac{1}{4+x^2}\,dx$, we make the substitution

$$\begin{aligned} x &= 2\tan(u), -\frac{\pi}{2} < u < \frac{\pi}{2}, \\ dx &= 2\sec^2(u)du. \end{aligned}$$

Then

$$\begin{aligned} \int \frac{1}{4+x^2}\,dx &= \int \frac{2sec^2(u)}{4+4\tan^2(u)}\,du \\ &= \frac{1}{2}\int \frac{\sec^2(u)}{1+\tan^2(u)}\,du \\ &= \frac{1}{2}\int \frac{\sec^2(u)}{\sec^2(u)}\,du \\ &= \frac{1}{2}\int du \\ &= \frac{1}{2}u + c. \end{aligned}$$

Since $x = 2\tan(u)$, $u = \tan^{-1}\left(\frac{x}{2}\right)$. Thus

$$\int \frac{1}{4+x^2}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c.$$

Example To evaluate $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx$, we make the substitution

$$\begin{aligned} x &= \tan(u), -\frac{\pi}{2} < u < \frac{\pi}{2}, \\ dx &= \sec^2(u)du. \end{aligned}$$

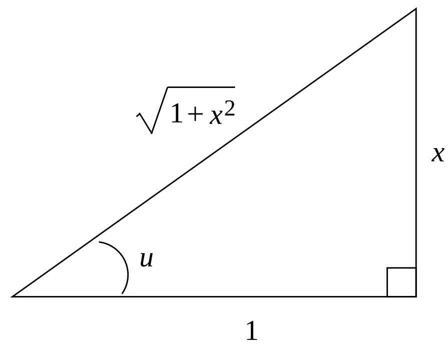

Figure 6.6.2 Right triangle with $\tan(u) = x$

Then

$$\sqrt{1+x^2} = \sqrt{1+\tan^2(u)} = \sqrt{\sec^2(u)} = |\sec(u)|.$$

Since $-\frac{\pi}{2} < u < \frac{\pi}{2}$, $\sec(u) > 0$, so $|\sec(u)| = \sec(u)$. Hence

$$\begin{aligned}\int \frac{1}{1+x^2}\,dx &= \int \frac{\sec^2(u)}{\sec(u)}\,du \\ &= \int \sec(u)du \\ &= \log|\sec(u)+\tan(u)| + c,\end{aligned}$$

where the final integral follows from an example in Section 6.2. Now $\tan(u) = x$, so

$$\sec^2(u) = 1 + \tan^2(u) = 1 + x^2.$$

Since $\sec(u) > 0$, it follows that

$$\sec(u) = \sqrt{1+x^2}.$$

Note that this expression for $\sec(u)$ may also be deduced from Figure 6.6.2, where we have a right triangle with an acute angle of size u such that $\tan(u) = x$. Thus

$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \log\left|\sqrt{1+x^2}+x\right| + c.$$

Secant substitutions

For integrals involving a factor of the form

$$\sqrt{x^2-a^2}$$

or

$$\frac{1}{\sqrt{x^2-a^2}},$$

where $a > 0$, the substitution

$$x = a\sec(u) \tag{6.6.6}$$

may be useful. With this substitution,

$$\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2(u) - a^2} = a\sqrt{\sec^2(u) - 1} = a\sqrt{\tan^2(u)} = a|\tan(u)|. \tag{6.6.7}$$

Now $\sqrt{x^2 - a^2}$ is meaningful only if either $x \geq a$ or $x \leq -a$. Since $x = a\sec(u)$, the former case corresponds to u in the interval $[0, \frac{\pi}{2})$ and the latter to u in the interval $(\frac{\pi}{2}, \pi]$. For $0 \leq u < \frac{\pi}{2}$, $\tan(u) \geq 0$, so

$$\sqrt{x^2 - a^2} = a\tan(u);$$

for $\frac{\pi}{2} < u \leq \pi$, $\tan(u) \leq 0$, so

$$\sqrt{x^2 - a^2} = -a\tan(u).$$

Hence it is important when evaluating integrals of this type to be careful about which values of x are of interest.

Example To evaluate $\displaystyle\int \frac{1}{\sqrt{x^2 - 9}}\,dx$ for $x > 3$, we make the substitution

$$\begin{aligned} x &= 3\sec(u), 0 < u < \frac{\pi}{2}, \\ dx &= 3\sec(u)\tan(u)du. \end{aligned}$$

Then

$$\begin{aligned} \int \frac{1}{\sqrt{x^2 - 9}}\,dx &= \int \frac{3\sec(u)\tan(u)}{\sqrt{9\sec^2(u) - 9}}\,du \\ &= \int \frac{3\sec(u)\tan(u)}{3\sqrt{\sec^2(u) - 1}}\,du \\ &= \int \frac{\sec(u)\tan(u)}{\sqrt{\tan^2(u)}}\,du \\ &= \int \frac{\sec(u)\tan(u)}{\tan(u)}\,du \\ &= \int \sec(u)du \\ &= \log|\sec(u) + \tan(u)| + c. \end{aligned}$$

Now $\sec(u) = \frac{x}{3}$, so

$$\tan^2(u) = \sec^2(u) - 1 = \frac{x^2}{9} - 1 = \frac{x^2 - 9}{9}.$$

Hence

$$\tan(u) = \frac{1}{3}\sqrt{x^2 - 9},$$

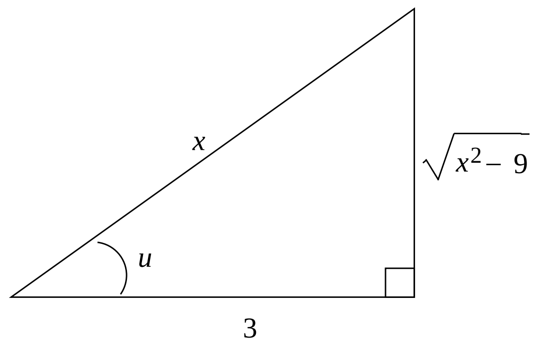

Figure 6.6.3 Right triangle with $\sec(u) = \frac{x}{3}$

where we have used the fact that $\tan(u) > 0$ since $0 < u < \frac{\pi}{2}$. Note that this expression for $\sec(u)$ may also be deduced from Figure 6.6.3, where we have a right triangle with an acute angle of size u such that $\sec(u) = \frac{x}{3}$. Thus

$$\int \frac{1}{\sqrt{x^2-9}}\,dx = \log\left|\frac{x}{3} + \frac{1}{3}\sqrt{x^2-9}\right| = \log|x + \sqrt{x^2-9}| - \log(3) + c.$$

Since $\log(3)$ is a constant, we may combine it with the arbitrary constant of integration. Moreover, since we are assuming $x > 3$, we may remove the absolute value and write

$$\int \frac{1}{\sqrt{x^2-9}}\,dx = \log(x + \sqrt{x^2-9}) + c.$$

Problems

1. Evaluate the following integrals.

   (a) $\displaystyle\int \frac{3}{\sqrt{16-x^2}}\,dx$

   (b) $\displaystyle\int \frac{x}{\sqrt{4-x^2}}\,dx$

   (c) $\displaystyle\int \sqrt{5-z^2}\,dz$

   (d) $\displaystyle\int \frac{5}{6+x^2}\,dx$

   (e) $\displaystyle\int \frac{1}{\sqrt{4+x^2}}\,dx$

   (f) $\displaystyle\int \frac{4x}{\sqrt{1+x^2}}\,dx$

2. Evaluate the following integrals.

   (a) $\displaystyle\int z\sqrt{1-z^2}\,dz$

   (b) $\displaystyle\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx$

   (c) $\displaystyle\int \frac{1}{\sqrt{x^2-4}}\,dx,\ x > 2$

   (d) $\displaystyle\int \frac{1}{\sqrt{x^2-4}}\,dx,\ x < -2$

   (e) $\displaystyle\int \frac{4}{\sqrt{3-2x^2}}\,dx$

   (f) $\displaystyle\int \frac{3}{5+2x^2}\,dx$

(g) $\displaystyle\int \sqrt{4-t^2}\,dt$

(h) $\displaystyle\int \sqrt{1-4x^2}\,dx$

3. Evaluate the following integrals.

   (a) $\displaystyle\int_0^3 \sqrt{9-t^2}\,dt$

   (b) $\displaystyle\int_{-1}^1 \sqrt{1+x^2}\,dx$

   (c) $\displaystyle\int_0^1 \frac{1}{(1+x^2)^{\frac{3}{2}}}\,dx$

   (d) $\displaystyle\int_0^1 x^2\sqrt{1-x^2}\,dx$

   (e) $\displaystyle\int_{5\sqrt{2}}^{10} \frac{1}{\sqrt{x^2-25}}\,dx$

   (f) $\displaystyle\int_{\sqrt{2}}^{2} \frac{3}{x^2\sqrt{x^2-1}}\,dx$

4. Evaluate
$$\int \frac{1}{1-x^2}\,dx$$
using (a) partial fractions and (b) the substitution $x = \sin(u)$. How do the two methods compare?

5. Evaluate
$$\int \frac{1}{\sqrt{1-x^2}}\,dx$$
using the substitution $x = \cos(u)$ with $0 < u < \pi$.

6. (a) Evaluate
$$\int \frac{1}{\sqrt{x^2-9}}\,dx$$
for $x < -3$.

   (b) Show that
$$\int \frac{1}{\sqrt{x^2-9}}\,dx = \log|x+\sqrt{x^2-9}| + c,$$
for both $x > 3$ and $x < -3$.