# THE LAWS OF SINES AND COSINES ON THE UNIT SPHERE AND HYPERBOLOID

DON HICKETHIER

Abstract. In a traditional trigonometry class the Law of Sines and Law of Cosines are fundamental tools used to solve triangles in a plane. Using vector analysis similar laws can be found for triangles on the unit sphere. With a slight alteration of the definition of "dot product" analogous laws are found for triangles on the unit hyperboloid.

## 1. Euclidean 3-space, $E^3$

**Definition 1.1.** Euclidean 3-space, $\mathbf{E}^3 = \left\{\mathbf{x} = \left(x^1, x^2, x^3\right) : x^1, x^2, x^3 \in \mathbb{R}\right\}$

**Definition 1.2** (Dot Product)**.** For $\mathbf{x}, \mathbf{y} \in \mathbf{E}^3$,

$$\begin{aligned} \mathbf{x}\cdot\mathbf{y} &= x^1y^1 + x^2y^2 + x^3y^3 \\ \mathbf{x}\cdot\mathbf{x} &= |\mathbf{x}|^2 = \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 \\ d_E\left(\mathbf{x},\mathbf{y}\right) &= |\mathbf{x}-\mathbf{y}| \end{aligned}$$

**Proposition 1.3.** *$d_E$ is a metric*

**Corollary 1.4** (Schwarz Inequality)**.** $|\mathbf{x}\cdot\mathbf{y}| \le |\mathbf{x}|\,|\mathbf{y}|$

Equality can be attained by including a multiple, $\cos\left(\theta\left(\mathbf{x},\mathbf{y}\right)\right)$, in the inequality.

$$\mathbf{x}\cdot\mathbf{y} = |\mathbf{x}|\,|\mathbf{y}|\cos\theta\left(\mathbf{x},\mathbf{y}\right)$$

where $\theta\left(\mathbf{x},\mathbf{y}\right)$ is the angle between $\mathbf{x}$ and $\mathbf{y}$.

**Definition 1.5** (Cross Product)**.** For $\mathbf{x}, \mathbf{y} \in \mathbf{E}^3$

$$\mathbf{x}\times\mathbf{y} = \begin{vmatrix} i & j & k \\ x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \end{vmatrix}$$

**Theorem 1.6.** *Properties of $\cdot$ and $\times$*

1. $\mathbf{x}\times\mathbf{y} = -\mathbf{y}\times\mathbf{x}$
2. $(\mathbf{x}\times\mathbf{y})\cdot\mathbf{z} = \begin{vmatrix} x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \\ z^1 & z^2 & z^3 \end{vmatrix}$

   $(\mathbf{x}\times\mathbf{y})\cdot\mathbf{z} = (\mathbf{z}\times\mathbf{x})\cdot\mathbf{y} = (\mathbf{y}\times\mathbf{z})\cdot\mathbf{x}$
3. $\mathbf{x}\times(\mathbf{y}\times\mathbf{z}) = (\mathbf{x}\cdot\mathbf{z})\,\mathbf{y} - (\mathbf{x}\cdot\mathbf{y})\,\mathbf{z}$
4. $(\mathbf{x}\times\mathbf{y})\cdot(\mathbf{z}\times\mathbf{w}) = \begin{vmatrix} \mathbf{x}\cdot\mathbf{z} & \mathbf{x}\cdot\mathbf{w} \\ \mathbf{y}\cdot\mathbf{z} & \mathbf{y}\cdot\mathbf{w} \end{vmatrix}$

---

Dave.

Property 4 combined with $\mathbf{x}\cdot\mathbf{y} = |\mathbf{x}|\,|\mathbf{y}|\cos\theta(\mathbf{x},\mathbf{y})$ yields

$$|\mathbf{x}\times\mathbf{y}| = |\mathbf{x}|\,|\mathbf{y}|\sin\theta(\mathbf{x},\mathbf{y})$$

1.1. **Triangles in $\mathbf{E}^3$.** Triangles in $\mathbf{E}^3$ consist of 3 points, $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbf{E}^3$ and the geodesics connecting the points.

*Geodesics* are "straight lines" between points. In $\mathbf{E}^3$, geodesics are straight lines.

| sides | $[\mathbf{x},\mathbf{y}]$ | $[\mathbf{y},\mathbf{z}]$ | $[\mathbf{z},\mathbf{x}]$ |
|---|---|---|---|
| lengths | $a = d_E(\mathbf{x},\mathbf{y})$ | $b = d_E(\mathbf{y},\mathbf{z})$ | $c = d_E(\mathbf{z},\mathbf{x})$ |
| angles | $\alpha$ | $\beta$ | $\gamma$ |

Law of Sines

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Law of Cosines

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$$

## 2. Unit Sphere

**Definition 2.1** (Unit Shpere, $\mathbf{S}^2$)**.** $\mathbf{S}^2 = \{\mathbf{x}\in\mathbf{E}^3 : |\mathbf{x}| = 1\}$

Notice $\mathbf{x}\cdot\mathbf{y} = |\mathbf{x}|\,|\mathbf{y}|\cos\theta(\mathbf{x},\mathbf{y}) = \cos\theta(\mathbf{x},\mathbf{y})$ and $|\mathbf{x}\times\mathbf{y}| = |\mathbf{x}|\,|\mathbf{y}|\sin\theta(\mathbf{x},\mathbf{y}) = \sin\theta(\mathbf{x},\mathbf{y})$.

On $\mathbf{S}^2$, the geodesic between two points is the shortest arc of the great circle passing through the points. This gives the distance between two points on the sphere to be

$$d_S(\mathbf{x},\mathbf{y}) = \theta(\mathbf{x},\mathbf{y}) = \cos^{-1}(\mathbf{x}\cdot\mathbf{y})$$

Then

$$\begin{aligned} 0 &\leq d_S(\mathbf{x},\mathbf{y}) \leq \pi \\ d_S(\mathbf{x},\mathbf{y}) &= \pi \Longleftrightarrow \mathbf{y} = -\mathbf{x} \text{ (antipodal)} \end{aligned}$$

**Proposition 2.2.** *$d_S$ is a metric.*

2.1. **Spherical Triangles.** Triangles in $\mathbf{S}^2$ consist of 3 points, $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbf{S}^2$ and the geodesics connecting the points.

| sides | $[\mathbf{x},\mathbf{y}]$ | $[\mathbf{y},\mathbf{z}]$ | $[\mathbf{z},\mathbf{x}]$ |
|---|---|---|---|
| lengths | $a = \theta(\mathbf{x},\mathbf{y})$ | $b = \theta(\mathbf{y},\mathbf{z})$ | $c = \theta(\mathbf{z},\mathbf{x})$ |
| angles | $\alpha$ | $\beta$ | $\gamma$ |
| geodesic | $\mathbf{f}:[0,a]\to\mathbf{S}^2$ | $\mathbf{g}:[0,b]\to\mathbf{S}^2$ | $\mathbf{h}:[0,c]\to\mathbf{S}^2$ |

Notice, using the right hand rule for cross product, that $\mathbf{z}\times\mathbf{x}$ is perpendicular to $\mathbf{h}'(0)$ and $\mathbf{y}\times\mathbf{z}$ is perpendicular to $-\mathbf{g}'(b)$. Now it is not difficult to observe

| | |
|---|---|
| $\theta(\mathbf{y}\times\mathbf{z},\mathbf{z}\times\mathbf{x}) = \pi-\alpha$ | $\theta(\mathbf{y}\times\mathbf{z},\mathbf{x}\times\mathbf{z}) = \alpha$ |
| $\theta(\mathbf{z}\times\mathbf{x},\mathbf{x}\times\mathbf{y}) = \pi-\beta$ | $\theta(\mathbf{z}\times\mathbf{x},\mathbf{y}\times\mathbf{x}) = \beta$ |
| $\theta(\mathbf{x}\times\mathbf{y},\mathbf{y}\times\mathbf{z}) = \pi-\gamma$ | $\theta(\mathbf{x}\times\mathbf{y},\mathbf{z}\times\mathbf{y}) = \gamma$ |

2.2. **Spherical Law of Sines.** Consider

$$\begin{aligned}(\mathbf{y}\times\mathbf{z})\times(\mathbf{z}\times\mathbf{x}) &= ((\mathbf{y}\times\mathbf{z})\cdot\mathbf{x})\,\mathbf{z}-((\mathbf{y}\times\mathbf{z})\cdot\mathbf{z})\,\mathbf{x}\\ &= ((\mathbf{y}\times\mathbf{z})\cdot\mathbf{x})\,\mathbf{z}\end{aligned}$$

Now take the norm of both sides

$$\begin{aligned}|(\mathbf{y}\times\mathbf{z})\times(\mathbf{z}\times\mathbf{x})| &= |((\mathbf{y}\times\mathbf{z})\cdot\mathbf{x})\,\mathbf{z}|\\ |\mathbf{y}\times\mathbf{z}|\,|\mathbf{z}\times\mathbf{x}|\sin\theta\,(\mathbf{y}\times\mathbf{z},\mathbf{z}\times\mathbf{x}) &= |((\mathbf{y}\times\mathbf{z})\cdot\mathbf{x})|\,|\mathbf{z}|\\ \sin b\sin c\sin(\pi-\alpha) &= |((\mathbf{y}\times\mathbf{z})\cdot\mathbf{x})|\\ \sin b\sin c\sin\alpha &= |((\mathbf{y}\times\mathbf{z})\cdot\mathbf{x})|\end{aligned}$$

Similarly

$$\begin{aligned}(\mathbf{z}\times\mathbf{x})\times(\mathbf{x}\times\mathbf{y}) &= ((\mathbf{z}\times\mathbf{x})\cdot\mathbf{y})\,\mathbf{x}\\ (\mathbf{x}\times\mathbf{y})\times(\mathbf{y}\times\mathbf{z}) &= ((\mathbf{x}\times\mathbf{y})\cdot\mathbf{z})\,\mathbf{y}\end{aligned}$$

Taking the norm of the reamining two equalities, noticing the right hand sides of each are equal, yields

$$\sin b\sin c\sin\alpha=\sin c\sin a\sin\beta=\sin a\sin b\sin\gamma$$

**Theorem 2.3.** *Spherical Law of Sines*

$$\frac{\sin b}{\sin\beta}=\frac{\sin a}{\sin\alpha}=\frac{\sin c}{\sin\gamma}$$

2.3. **Spherical Law of Cosines.** Consider

$$\begin{aligned}(\mathbf{y}\times\mathbf{z})\cdot(\mathbf{z}\times\mathbf{x}) &= \begin{vmatrix}\mathbf{y}\cdot\mathbf{z} & \mathbf{y}\cdot\mathbf{x}\\ \mathbf{z}\cdot\mathbf{z} & \mathbf{z}\cdot\mathbf{x}\end{vmatrix}\\ &= (\mathbf{y}\cdot\mathbf{z})\,(\mathbf{z}\cdot\mathbf{x})-(\mathbf{y}\cdot\mathbf{x})\,(\mathbf{z}\cdot\mathbf{z})\\ &= (\mathbf{y}\cdot\mathbf{z})\,(\mathbf{z}\cdot\mathbf{x})-(\mathbf{y}\cdot\mathbf{x})\end{aligned}$$

Using $\mathbf{u}\cdot\mathbf{v}=|\mathbf{u}|\,|\mathbf{v}|\cos\theta\,(\mathbf{u},\mathbf{v})$ gives

$$|\mathbf{y}\times\mathbf{z}|\,|\mathbf{z}\times\mathbf{x}|\cos\theta\,(\mathbf{y}\times\mathbf{z},\mathbf{z}\times\mathbf{x})$$
$$=\cos\theta\,(\mathbf{y},\mathbf{z})\cos\theta\,(\mathbf{z},\mathbf{x})-\cos\theta\,(\mathbf{x},\mathbf{y})$$

Furthermore, $|\mathbf{u}\times\mathbf{v}|=|\mathbf{u}|\,|\mathbf{v}|\sin\theta(\mathbf{u},\mathbf{v})$, so

$$\begin{aligned}\sin b\sin c\cos(\pi-\alpha) &= \cos b\cos c-\cos a\\ -\cos\alpha &= \frac{\cos b\cos c-\cos a}{\sin b\sin c}\\ \cos\alpha &= \frac{\cos a-\cos b\cos c}{\sin b\sin c}\end{aligned}$$

## 3. Minkowski 3-space

**Definition 3.1** (Minkowski 3-space, $\mathbf{M}^3$)**.** $\mathbf{M}^3=\left\{\mathbf{x}:\mathbf{x}=\left(x^1,x^2,x^3\right)\right\}$

**Definition 3.2** (BoxDot Product)**.** For $\mathbf{x},\mathbf{y}\in\mathbf{M}^3$,

$$\begin{aligned}\mathbf{x}\boxdot\mathbf{y} &= x^1y^1+x^2y^2-x^3y^3\\ \mathbf{x}\boxdot\mathbf{x} &= \|\mathbf{x}\|^2\\ d_L(\mathbf{x},\mathbf{y}) &= \|\mathbf{x}-\mathbf{y}\|\end{aligned}$$

$\mathbf{x} \in \mathbf{M}^3$ is called *time-like* if $\mathbf{x} \boxdot \mathbf{x} < 0$.
$\mathbf{x} \in \mathbf{M}^3$ is called *space-like* if $\mathbf{x} \boxdot \mathbf{x} > 0$.
$\mathbf{x} \in \mathbf{M}^3$ is called *light-like* if $\mathbf{x} \boxdot \mathbf{x} = 0$.
We will be mainly concerned with time-like vectors for the remainder of the time.

**Proposition 3.3.** *$d_L$ is not a metric*

**Corollary 3.4.** *For $\mathbf{x}$ and $\mathbf{y}$ timelike vectors,*
$\mathbf{x} \boxdot \mathbf{y} \geq \|\mathbf{x}\| \|\mathbf{y}\|$

Equality can be attained by including a multiple, $\cosh(\theta(\mathbf{x}, \mathbf{y}))$, in the inequality.

$$\mathbf{x} \boxdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cosh\theta(\mathbf{x}, \mathbf{y})$$

where $\theta(\mathbf{x}, \mathbf{y})$ is the hyperbolic angle between $\mathbf{x}$ and $\mathbf{y}$.

**Definition 3.5** (BoxCross Product)**.** For $\mathbf{x}, \mathbf{y} \in \mathbf{M}^3$

$$\mathbf{x} \boxtimes \mathbf{y} = \begin{vmatrix} i & j & -k \\ x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \end{vmatrix}$$

**Theorem 3.6.** *Properties of Vectors in Minkowski 3-Space*

1. *If $x$, $y$ are positive time-like vectors, then $x \boxtimes y$ is space-like.*
2. *If $u$, $v$ are space-like vectors, then the following are equivalent:*
   (a) *The vectors $u$ and $v$ satisfy the inequality*
   $|u \boxdot v| < \|u\| \|v\|$.
   (b) *$u \boxtimes v$ is time-like.*
   (c) *The vector subspace $V$ spanned by $u$ and $v$ is space-like (every nonzero vector is space-like).*
3. *If $u$, $v$ are space-like vectors spanning a space-like vector space, then*

$$\begin{aligned} u \boxdot v &= \|u\| \|v\| \cos\theta(u, v) \\ |\|u \boxtimes v\|| &= \|u\| \|v\| \sin\theta(u, v) \end{aligned}$$

   *where* $|\|u\||^2 = -(u \boxdot u)$.

**Theorem 3.7.** *Properties of $\boxdot$ and $\boxtimes$*

1. $\mathbf{x} \boxtimes \mathbf{y} = -\mathbf{y} \boxtimes \mathbf{x}$
2. $(\mathbf{x} \boxtimes \mathbf{y}) \boxdot \mathbf{z} = \begin{vmatrix} x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \\ z^1 & z^2 & z^3 \end{vmatrix}$
3. $\mathbf{x} \boxtimes (\mathbf{y} \boxtimes \mathbf{z}) = -((\mathbf{x} \boxdot \mathbf{z})\mathbf{y} - (\mathbf{x} \boxdot \mathbf{y})\mathbf{z})$
4. $(\mathbf{x} \boxtimes \mathbf{y}) \boxdot (\mathbf{z} \boxtimes \mathbf{w}) = -\begin{vmatrix} \mathbf{x} \boxdot \mathbf{z} & \mathbf{x} \boxdot \mathbf{w} \\ \mathbf{y} \boxdot \mathbf{z} & \mathbf{y} \boxdot \mathbf{w} \end{vmatrix}$

For $\mathbf{x}$ and $\mathbf{y}$ time-like vectors, property 4 combined with $\mathbf{x} \boxdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cosh\theta(\mathbf{x}, \mathbf{y})$ yields

$$\|\mathbf{x} \times \mathbf{y}\| = -\|\mathbf{x}\| \|\mathbf{y}\| \sinh\theta(\mathbf{x}, \mathbf{y})$$

3.1. **Unit Hyperboloid, $\mathbf{H}^2$.**

**Definition 3.8.** $\mathbf{H}^2 = \{\mathbf{x} \in \mathbf{M}^3 : \mathbf{x} \boxdot \mathbf{x} = -1\}$

Notice $\mathbf{x} \boxdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cosh\theta(\mathbf{x}, \mathbf{y}) = -\cosh\theta(\mathbf{x}, \mathbf{y})$ and $\|\mathbf{x} \boxtimes \mathbf{y}\| = -\|\mathbf{x}\| \|\mathbf{y}\| \sinh\theta(\mathbf{x}, \mathbf{y}) = \sinh\theta(\mathbf{x}, \mathbf{y})$.

On $\mathbf{H}^2$, the geodesic is the branch of a hyperbola passing through the points. This gives the distance between two points on the hyperboloid to be

$$d_H(\mathbf{x}, \mathbf{y}) = \theta(\mathbf{x}, \mathbf{y}) = \cosh^{-1}(-\mathbf{x} \boxdot \mathbf{y})$$

Notice

$$0 \leq d_H(\mathbf{x}, \mathbf{y})$$

**Proposition 3.9.** *$d_H$ is a metric.*

3.2. **Hyperbolic Triangles.** Triangles on $\mathbf{H}^2$ consist of 3 points, $\mathbf{x}, \mathbf{y}, \mathbf{z}$, and the geodesics connecting the points.

| sides | $[\mathbf{x}, \mathbf{y}]$ | $[\mathbf{y}, \mathbf{z}]$ | $[\mathbf{z}, \mathbf{x}]$ |
|---|---|---|---|
| lengths | $a = \theta(\mathbf{x}, \mathbf{y})$ | $b = \theta(\mathbf{y}, \mathbf{z})$ | $c = \theta(\mathbf{z}, \mathbf{x})$ |
| angles | $\alpha$ | $\beta$ | $\gamma$ |
| geodesic | $\mathbf{f} : [0, a] \to \mathbf{H}^2$ | $\mathbf{g} : [0, b] \to \mathbf{H}^2$ | $\mathbf{h} : [0, c] \to \mathbf{H}^2$ |

$\mathbf{h}'(0)$ and $-\mathbf{g}'(b)$ are space-like vectors spanning a space-like vector space. This implies the angles are space-like angles, that is they are measured using a protractor.

| | |
|---|---|
| $\theta(\mathbf{y} \boxtimes \mathbf{z}, \mathbf{z} \boxtimes \mathbf{x}) = \pi - \alpha$ | $\theta(\mathbf{y} \boxtimes \mathbf{z}, \mathbf{x} \boxtimes \mathbf{z}) = \alpha$ |
| $\theta(\mathbf{z} \boxtimes \mathbf{x}, \mathbf{x} \boxtimes \mathbf{y}) = \pi - \beta$ | $\theta(\mathbf{z} \boxtimes \mathbf{x}, \mathbf{y} \boxtimes \mathbf{x}) = \beta$ |
| $\theta(\mathbf{x} \boxtimes \mathbf{y}, \mathbf{y} \boxtimes \mathbf{z}) = \pi - \gamma$ | $\theta(\mathbf{x} \boxtimes \mathbf{y}, \mathbf{z} \boxtimes \mathbf{y}) = \gamma$ |

3.3. **Hyperbolic Law of Sines.** Consider

$$\begin{aligned}(\mathbf{y} \boxtimes \mathbf{z}) \boxtimes (\mathbf{z} \boxtimes \mathbf{x}) &= -(((\mathbf{y} \boxtimes \mathbf{z}) \boxdot \mathbf{x})\,\mathbf{z} - ((\mathbf{y} \boxtimes \mathbf{z}) \boxdot \mathbf{z})\,\mathbf{x}) \\ &= -((\mathbf{y} \boxtimes \mathbf{z}) \boxdot \mathbf{x})\,\mathbf{z}\end{aligned}$$

Now take the norm of both sides, where $\mathbf{y} \boxtimes \mathbf{z}$ and $\mathbf{z} \boxtimes \mathbf{x}$ are spacelike vectors

$$|\|(\mathbf{y} \boxtimes \mathbf{z}) \boxtimes (\mathbf{z} \boxtimes \mathbf{x})\|| = |\|-((\mathbf{y} \boxtimes \mathbf{z}) \boxdot \mathbf{x})\,\mathbf{z}\||$$

$$\|\mathbf{y} \boxtimes \mathbf{z}\| \|\mathbf{z} \boxtimes \mathbf{x}\| \sin\theta(\mathbf{y} \boxtimes \mathbf{z}, \mathbf{z} \boxtimes \mathbf{x}) = |-((\mathbf{y} \boxtimes \mathbf{z}) \boxdot \mathbf{x})| \; |\|\mathbf{z}\||$$

$$\begin{aligned}\sinh b \sinh c \sin(\pi - \alpha) &= |((\mathbf{y} \boxtimes \mathbf{z}) \boxdot \mathbf{x})|\,(-\mathbf{z} \boxdot \mathbf{z}) \\ \sinh b \sinh c \sin\alpha &= |((\mathbf{y} \boxtimes \mathbf{z}) \boxdot \mathbf{x})|\end{aligned}$$

Similarly

$$\begin{aligned}(\mathbf{z} \boxtimes \mathbf{x}) \boxtimes (\mathbf{x} \boxtimes \mathbf{y}) &= -((\mathbf{z} \boxtimes \mathbf{x}) \boxdot \mathbf{y})\,\mathbf{x} \\ (\mathbf{x} \boxtimes \mathbf{y}) \boxtimes (\mathbf{y} \boxtimes \mathbf{z}) &= -((\mathbf{x} \boxtimes \mathbf{y}) \boxdot \mathbf{z})\,\mathbf{y}\end{aligned}$$

Taking the norm of the remaining two equalities, noticing the right hand sides of each are equal, yields

$$\sinh b \sinh c \sin\alpha = \sinh c \sinh a \sin\beta = \sinh a \sinh b \sin\gamma$$

**Theorem 3.10.** *Hyperbolic Law of Sines*

$$\frac{\sinh b}{\sin \beta} = \frac{\sinh a}{\sin \alpha} = \frac{\sinh c}{\sin \gamma}$$

3.4. **Hyperbolic Law of Cosines.** Consider

$$\begin{aligned}(\mathbf{y} \boxtimes \mathbf{z}) \boxdot (\mathbf{z} \boxtimes \mathbf{x}) &= -\begin{vmatrix} \mathbf{y} \boxdot \mathbf{z} & \mathbf{y} \boxdot \mathbf{x} \\ \mathbf{z} \boxdot \mathbf{z} & \mathbf{z} \boxdot \mathbf{x} \end{vmatrix} \\ &= -\left((\mathbf{y} \boxdot \mathbf{z})(\mathbf{z} \boxdot \mathbf{x}) - (\mathbf{y} \boxdot \mathbf{x})(-\mathbf{1})\right) \\ &= -\left((\mathbf{y} \boxdot \mathbf{z})(\mathbf{z} \boxdot \mathbf{x}) + (\mathbf{y} \boxdot \mathbf{x})\right)\end{aligned}$$

Using $\mathbf{u} \boxdot \mathbf{v} = -\cosh \theta (\mathbf{u}, \mathbf{v})$ for $\mathbf{u}$, $\mathbf{v} \in \mathbf{H}^2$ and $\mathbf{u} \boxdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta (\mathbf{u}, \mathbf{v})$ for $\mathbf{u}$, $\mathbf{v}$ space-like gives

$$\|\mathbf{y} \boxtimes \mathbf{z}\| \|\mathbf{z} \boxtimes \mathbf{x}\| \cos \theta (\mathbf{y} \boxtimes \mathbf{z}, \mathbf{z} \boxtimes \mathbf{x}) = -\left(\cosh \theta (\mathbf{y}, \mathbf{z}) \cosh \theta (\mathbf{z}, \mathbf{x}) - \cosh \theta (\mathbf{x}, \mathbf{y})\right)$$

Furthermore, $\|\mathbf{u} \boxtimes \mathbf{v}\| = \sinh \theta(\mathbf{u}, \mathbf{v})$ for $\mathbf{u}$, $\mathbf{v} \in \mathbf{H}^2$, so

$$\begin{aligned}\sinh b \sinh c \cos(\pi - \alpha)) &= -(\cosh b \cosh c - \cosh a) \\ -\cos \alpha &= \frac{-(\cosh b \cosh c - \cosh a)}{\sinh b \sinh c} \\ \cos \alpha &= \frac{\cosh b \cosh c - \cosh a}{\sinh b \sinh c}\end{aligned}$$

College of the Redwoods
*E-mail address*: Don-Hickethier@Eureka.Redwoods.cc.ca.us